1 May 2001, Mechanics, Problem 1

1.1 (a)

Take a segment of length dx and draw its free-body diagram. For the segment to be static we need the condition (with x increasing downward):

$$T(x+dx) + mg = T(x)$$
$$T(x+dx) - T(x) = -\rho dxg$$
$$\frac{dT}{dx} = -\rho g$$

Now let's look internally at the piece of string for which we wrote this equation. It has ends at x and x+dx, which stretch by amounts s(x) and s(x+dx), respectively. The tension in the string will be:

$$T = k_{dx}[s(x+dx) - s(x)]$$

We need to find the k_{dx} of this little piece of spring. If we stretch a spring of natural length L and spring constant k by a given amount b, it is equivalent to stretching each little piece of length dx by an amount b dx/L. Thus, we can write:

$$Kb = k_{dx}bdx/L$$
$$KL = k_{dx}dx$$
$$T = \frac{KL}{dx}[s(x+dx) - s(x)] = \frac{KL}{dx}ds = KL\frac{\partial s}{\partial x}$$
$$KL\frac{\partial^2 s}{\partial x^2} = -\rho g$$
$$s(x) = -\frac{\rho g}{2KL}x^2 + Ax + B$$

Using the boundary conditions:

$$T(L) = 0$$
$$s(0) = 0$$

$$s(x) = -\frac{\rho g}{2KL}x^2 + \frac{\rho g}{K}x \tag{1}$$

1.2 (b)

The equation, including a possibility for movement, is:

$$T(x+dx) + mg - T(x) = m\frac{d^2s}{dt^2}$$

$$KL\frac{\partial^2 s}{\partial x^2} + \rho g = \rho\frac{d^2s}{dt^2}$$

When we turn off gravity, we will get:

$$KL\frac{\partial^2 s}{\partial x^2} = \rho\frac{d^2s}{dt^2}$$

The most general solution is in terms of two completely arbitrary functions f and g:

$$s(x,t) = f\left(x + \sqrt{\frac{KL}{\rho}}t\right) + g\left(x - \sqrt{\frac{KL}{\rho}}t\right)$$

$$s(x,0) = f(x) + g(x) = -\frac{\rho g}{2KL}x^2 + \frac{\rho g}{K}x$$

$$\dot{s}(x,0) = \sqrt{\frac{KL}{\rho}}[f'(x) - g'(x)] = 0$$

$$f(x) = -\frac{\rho g}{4KL}x^2 + \frac{\rho g}{2K}x + \frac{C}{2}$$

$$g(x) = -\frac{\rho g}{4KL}x^2 + \frac{\rho g}{2K}x - \frac{C}{2}$$

where C is some arbitrary constant. Then we get:

$$s(x,t) = -\frac{\rho g}{4KL}\left(x + \sqrt{\frac{KL}{\rho}}t\right)^2 + \frac{\rho g x}{K} - \frac{\rho g}{4KL}\left(x - \sqrt{\frac{KL}{\rho}}t\right)^2 \tag{2}$$