1 May 2001, Mechanics, Problem 1

1.1 (a)

Take a segment of length dx and draw its free-body diagram. For the segment to be static we need the condition (with x increasing downward):

\[ T(x + dx) + mg = T(x) \]
\[ T(x + dx) - T(x) = -\rho dx g \]
\[ \frac{dT}{dx} = -\rho g \]

Now let’s look internally at the piece of string for which we wrote this equation. It has ends at x and x+dx, which stretch by amounts s(x) and s(x+dx), respectively. The tension in the string will be:

\[ T = k_{dx}[s(x + dx) - s(x)] \]

We need to find the \( k_{dx} \) of this little piece of spring. If we stretch a spring of natural length L and spring constant k by a given amount b, it is equivalent to stretching each little piece of length dx by an amount \( b \ dx/L \). Thus, we can write:

\[ Kb = k_{dx}b dx/L \]
\[ KL = k_{dx} dx \]
\[ T = \frac{KL}{dx}[s(x + dx) - s(x)] = \frac{KL}{dx} ds = KL \frac{\partial s}{\partial x} \]
\[ KL \frac{\partial^2 s}{\partial x^2} = -\rho g \]
\[ s(x) = -\frac{\rho g}{2KL} x^2 + Ax + B \]

Using the boundary conditions:

\[ T(L) = 0 \]
\[ s(0) = 0 \]

\[ s(x) = -\frac{\rho g}{2KL} x^2 + \frac{\rho g}{K} \]

(1)
1.2 (b)

The equation, including a possibility for movement, is:

\[
T(x + dx) + mg - T(x) = m \frac{d^2 s}{dt^2}
\]

\[
KL \frac{\partial^2 s}{\partial x^2} + \rho g = \rho \frac{d^2 s}{dt^2}
\]

When we turn off gravity, we will get:

\[
KL \frac{\partial^2 s}{\partial x^2} = \rho \frac{d^2 s}{dt^2}
\]

The most general solution is in terms of two completely arbitrary functions \(f\) and \(g\):

\[
s(x, t) = f \left( x + \sqrt{\frac{KL}{\rho} t} \right) + g \left( x - \sqrt{\frac{KL}{\rho} t} \right)
\]

\[
s(x, 0) = f(x) + g(x) = -\frac{\rho g}{2KL} x^2 + \frac{\rho g}{K} x
\]

\[
\dot{s}(x, 0) = \sqrt{\frac{KL}{\rho}} \left[ f'(x) - g'(x) \right] = 0
\]

\[
f(x) = -\frac{\rho g}{4KL} x^2 + \frac{\rho g}{2K} x + \frac{C}{2}
\]

\[
g(x) = -\frac{\rho g}{4KL} x^2 + \frac{\rho g}{2K} x - \frac{C}{2}
\]

where \(C\) is some arbitrary constant. Then we get:

\[
s(x, t) = -\frac{\rho g}{4KL} \left( x + \sqrt{\frac{KL}{\rho} t} \right)^2 + \frac{\rho g x}{K} - \frac{\rho g}{4KL} \left( x - \sqrt{\frac{KL}{\rho} t} \right)^2
\]

\[(2)\]