M01M.1

A spring of spring constant K, unstretched length L, and mass per unit length ρ. The spring is suspended vertically from one end in a constant gravitational field, g, and stretches under its own weight.

(a) For a point whose distance from the upper end of the spring is x when unstretched, find its distance $s(x)$ from its gravity-free position when the spring is stretched.

(b) Suppose we suddenly "turn off" gravity. Find the subsequent motion $s(x,t)$ of the spring.

(a) Balancing the forces on a small part dx of the spring gives:

$$T(x+dx)+\rho g dx-T(x)=0$$

$$\frac{dT}{dx}=-\rho g$$

$$T=k_{dx}[s(x+dx)-s(x)],$$

where k_{dx} is the spring constant for the dx portion of the spring and is related to the spring constant K by $KL=k_{dx}dx$. Thus,

$$T=KL\frac{ds}{dx}$$

$$KL\frac{d^2s}{dx^2}=-\rho g$$

Using the boundary conditions $T(x=L)=0$ since there is no tension at the end of the spring and $s(x=0)=0$ since the spring is attached to the wall, we get

$$s(x)=-\frac{\rho g}{2KL}x^2+\frac{\rho g}{K}x.$$

(b) Including the time dependence of s, from (a) we get

$$KL\frac{d^2s}{dx^2} + \rho g = \rho\frac{d^2s}{dt^2}\,.$$

By setting gravity to zero, we get the wave equation with $v = \sqrt{KL/\rho}$,

$$\frac{d^2s}{dx^2} = \frac{\rho}{KL}\frac{d^2s}{dt^2}\,.$$

The general solution to the wave equation is $s(x,t) = f(x-vt) + g(x+vt)$. Using the boundary conditions,

$$s(x,t=0) = f(x) + g(x) = -\frac{\rho g}{2KL}x^2 + \frac{\rho g}{K}x$$

$$\frac{ds}{dt}(x,t=0) = -vf'(x) + vg'(x) = 0$$

We thus obtain equations for $f(x)$ and $g(x)$

$$f(x) = -\frac{\rho g}{4KL}x^2 + \frac{\rho g}{2K}x + C/2$$

$$g(x) = -\frac{\rho g}{4KL}x^2 + \frac{\rho g}{2K}x - C/2$$

Putting in the time dependence, we get

$$s(x,t) = -\frac{\rho g}{4KL}(x-vt)^2 - \frac{\rho g}{4KL}(x+vt)^2 + \frac{\rho g}{K}x$$

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2 thoughts on “M01M.1”

December 4, 2013 at 3:19 am

Actually, after some time, I realized that your $s(x,t)$ doesn't satisfy boundary conditions. Think why and how to fix this. Later I can give a hint if needed.

October 3, 2013 at 6:16 pm

Very good!