1 May 2001, Electromagnetism, Problem 3

1.1 (a)

Since I can never remember the solutions to these problems, I have to solve it every time. We want to solve Laplace's equation everywhere in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{\partial^2 V}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

There is a complete angular symmetry, so the potential cannot depend on ϕ, and we therefore throw out that term. Also, since we are told that the cylinders are long and thin, we will take them to be approximately infinite, such that the potential is independent of z:

$$V = A\log(\rho) + B$$

The physics is invariant under a constant shift of the potential, so I will take B=0. Also, in the region inside the inner cylinder, the potential needs to remain finite even in the center, so we need to set $A_{in} = 0$. This gives us:

$$\boldsymbol{E}_{out} = -\frac{A_{out}}{\rho}\hat{\rho}$$

$$\boldsymbol{E}_{bet} = -\frac{A_{bet}}{\rho}\hat{\rho}$$

$$\boldsymbol{E}_{in} = \boldsymbol{0}$$

Use the boundary condition that the change in the perpendicular component of the electric field is given by the surface charge density:

$$A_{bet} = -\frac{a\sigma_a}{\epsilon_0}$$

$$-\frac{A_{out}}{b} + \frac{A_{bet}}{b} = \frac{\sigma_b}{\epsilon_0}$$

$$\boldsymbol{E}_{out} = \frac{a\sigma_a + b\sigma_b}{\rho\epsilon_0}\hat{\rho} \quad (1)$$

$$\boldsymbol{E}_{bet} = \frac{a\sigma_a}{\rho\epsilon_0}\hat{\rho} \quad (2)$$

$$\boldsymbol{E}_{in} = \boldsymbol{0} \quad (3)$$

1.2 (b)

$$a\sigma_a + b\sigma_b = 0 \quad (4)$$

1.3 (c)

This is entirely analogous to the case of a solenoid, with surface current nI, which generates an electric field inside the cylinder $\mu_0 nI$, and no magnetic field outside the cylinder. In our case the surface current is:

$$\boldsymbol{K} = \sigma \boldsymbol{v} = \sigma_b b \omega_b \hat{\phi}$$
$$\boldsymbol{B} = \mu_0 \sigma_b b \omega_b \hat{z} \tag{5}$$

1.4 (d)

The magnetic field inside both cylinders will have a contribution from each rotation:

$$\boldsymbol{B} = (\mu_0 \sigma_b b \omega_b + \mu_0 \sigma_a a \omega_a) \hat{z}$$

In order for this to be 0, using the relation found in part b, we need:

$$\omega_a = \omega_b \tag{6}$$

1.5 (e)

The integral form of Faraday's law is:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$$

where the flux in the right hand side is over the area enclosed by the path on the left hand side. If we compute the induced electric field around the inner cylinder, we will get 0, because the magnetic field inside the inner cylinder is 0. The induced electric field in the outer cylinder is given by:

$$E 2\pi b = -\pi(b^2 - a^2)\frac{\partial B}{\partial t} = -\pi(b^2 - a^2)\mu_0 \sigma_b b \dot{\omega}_b$$
$$\boldsymbol{E} = -\frac{(b^2 - a^2)}{2}\mu_0 \sigma_b \dot{\omega}_b \hat{\phi}$$

The charges on the cylinder experience a force:

$$\boldsymbol{F} = -\pi b l (b^2 - a^2)\mu_0 \sigma_b^2 \dot{\omega}_b \hat{\phi}$$

The torque needed to oppose this force is:

$$\boldsymbol{\tau} = -b\hat{\rho} \times \boldsymbol{F} = \pi b^2 l (b^2 - a^2)\mu_0 \sigma_b^2 \dot{\omega}_b \hat{z} \tag{7}$$

1.6 (f)

$$\boldsymbol{E} \times \boldsymbol{B} = \frac{(\sigma_b b)^2 \mu_0 \omega_b}{\rho \epsilon_0} \hat{\phi}$$

$$\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) = \frac{(\sigma_b b)^2 \mu_0 \omega_b}{\rho \epsilon_0} (\rho \hat{z} - z \hat{\rho})$$

$$\boldsymbol{L}_{EM} = (\sigma_b b)^2 \mu_0 \omega_b l 2\pi \int_a^b \rho \, d\rho \hat{z} = (\sigma_b b)^2 \mu_0 \omega_b l (b^2 - a^2) \pi \hat{z} \tag{8}$$

which is equal to the time integration of the torque found in part (e).