M01E.2

a) From Maxwell's equations and Ohm's law, we quickly arrive at the relation $k^2 = i\omega\mu\sigma + \omega^2\epsilon\mu$. The task is to evaluate $\sigma(\omega)$ for this case. Assuming the electrons are free, we have the equation of motion

$$m\dot{v} = -eE_o e^{-i\omega t}$$

Which has solution $v = \frac{eE_o}{im\omega}e^{-i\omega t}$. We can write the current density $J = -n_e eV = -\frac{ne^2 E_o}{im\omega}e^{-i\omega t}$ and consequently identify

$$\sigma(\omega) = -\frac{ne^2}{im\omega}.$$

Substituting this into our equation we find that

$$k = \omega\sqrt{\mu_o\epsilon_o\left(1 - \frac{\omega_p^2}{\omega^2}\right)}$$

Where we identify $\omega_p^2 = \frac{ne^2}{m_e}$.

b) $\frac{m_e}{m_p} \approx 5.4 \times 10^{-4}$. So $\frac{\omega_{p,e}^2}{\omega_{p,p}^2} \approx 10^4$. This means that for frequencies ω above both plasma frequencies, we can generally take $\omega_{p,p} << \omega$ since $\omega_{p,e}$ is 100 times larger. We can thus generally ignore dispersive effects due to the nuclei.

c) We only care about the case when $\omega > \omega_p$. Below that frequency our solutions are exponentially decaying functions, not waves properly speaking.

$$v_{\text{phase}} = k/\omega = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$ which is larger than c.

$v_{\text{group}} = \frac{d\omega}{dk} = 1/(\frac{dk}{d\omega}) = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}.$ Which is always less than c.

d) The Vela pulsar is $= 3 \times 10^{18}$ cm away. It emits radio waves over a broad band. Narrow pulses with a fixed period are sent ou at different frequencies. $f_1 = 1660$MHz pulses are delayed relative to $f_2 = 1720$MHz pulses by 6.8ms. What is the mean density of free electrons between Vela and us?

Since $v = d/t$ it follows that $1/v_1 - 1/v_2 = \Delta t/d$. Anticipating that $\omega_p << \omega$ we take the group velocity (since this are actually narrow bandwidth frequency packets) to be

$$\frac{1}{v_{grp}} = \frac{1}{c}\left(1 + \frac{\omega_p^2}{2\omega^2}\right).$$

Which gives

$$n = \frac{4\pi^2 c\Delta t}{d}\,\frac{f_1^2 f_2^2}{f_2^2 - f_1^2}.$$

Plugging in numerical values,

$$n = \frac{(4\pi^2)(3\times10^{10}\ \text{cm/s})(6.8\times10^{-3}\ \text{s})}{3\times10^{18}\ \text{cm}}\,\frac{1660^2\times1720^2\times10^{24}\ \text{Hz}^2}{(1720^2-1660^2)*10^{12}\ \text{Hz}} = 1.08\times10^{11}\ \frac{\text{electrons}}{\text{cm}^3}.$$

One thought on “M01E.2”

December 13, 2013 at 6:42 pm

Your solution went quite well until the end of (d).

Redo your calculation of n. Actually, the formula for n doesn't look right.