M01E.2

a)

For electromagnetic plane waves, we begin by taking the curl of Faraday's Law,

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

Substituting Ampere's law into the above equation for $\mathbf{B}$ gives

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2}$$

For an electromagnetic plane wave, we assume a perturbation field of the form

$$\mathbf{E} = E_0 e^{-i(\omega t + \mathbf{k} \cdot \mathbf{r})}\ \hat{\mathbf{n}}. \tag{3}$$

Likewise, current (or particle velocity) also has this harmonic dependence. Substituting this into equation 2 results in

$$k(\mathrm{k} \cdot \mathbf{E}) - k^2 \mathbf{E} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \omega^2 \mathbf{E} \tag{4}$$

Note that in this equation $\mathbf{J}$ denotes the perturbation current. The fluid equation for momentum in electromagnetic fields is given by

$$mn \frac{\partial \mathbf{v}}{\partial t} = e^2 n \mathbf{E} \tag{5}$$

We can linearize n and $\mathbf{v}$, giving

$$n = n_0 + n_1 \tag{6}$$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \tag{7}$$

where the subscript denotes order and $\mathbf{v}_0 = 0$. Linearize the momentum equation. keeping first order terms, results in

$$mn_0 \frac{\partial \mathbf{v}_1}{\partial t} = e^2 n_0 \mathbf{E}. \tag{8}$$

where we note $\mathbf{E}$ is itself first order.
Multiplying this by e gives

$$m \frac{\partial \mathbf{J}}{\partial t} = e^2 n_0 \mathbf{E}. \tag{9}$$

Substituting this equation into 4 gives

$$k(\mathrm{k} \cdot \mathbf{E}) - k^2 \mathbf{E} = \frac{\mu_0 e^2 n_0}{m} \mathbf{E} - \frac{\omega^2}{c^2} \mathbf{E} = \frac{\omega_\mathrm{p}^2}{c^2} \mathbf{E} - \frac{\omega^2}{c^2} \mathbf{E} \tag{10}$$

Since we're dealing with plane waves, $\mathbf{E}$ is perpendicular to $\mathbf{k}$, giving

$$-c^2 k^2 \mathbf{E} = \omega_\mathrm{p}^2 \mathbf{E} - \omega^2 \mathbf{E} \tag{11}$$

This leads to the dispersion relation

$$\omega^2 = \omega_\mathrm{p}^2 + c^2 k^2 \tag{12}$$

where

$$\omega_\mathrm{p}^2 = \frac{n_0 e^2}{m \epsilon_0}. \tag{13}$$

b)

As the plasma frequency is proportional to the inverse root of the particle mass, ion contributions can be neglected when the wave frequency is larger than the electron plasma frequency.

c)

Rewriting the dispersion relation,

$$k = c^{-1}\sqrt{\omega^2 - \omega_{\rm p}^2} \tag{14}$$

The phase velocity is given by

$$v_{\rm ph} = \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - \omega_{\rm p}^2}} = \frac{c}{\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}}} \tag{15}$$

while the group velocity is given by

$$v_{\rm g} = \frac{\partial \omega}{\partial k} = c\sqrt{\omega^2 - \omega_{\rm p}^2} = c\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}} \tag{16}$$

d)

The change in time is given by

$$\Delta t = t_1 - t_2 = d(\frac{1}{v_1} - \frac{1}{v_2}) = \frac{d}{c}\left(\left(1 - \frac{\omega_{\rm p}^2}{\omega_1^2}\right)^{-1/2} - \left(1 - \frac{\omega_{\rm p}^2}{\omega_2^2}\right)^{-1/2}\right) \tag{17}$$

which to first order results in

$$\omega_{\rm p}^2\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) = \frac{2c\Delta t}{d} \tag{18}$$

or

$$\omega_{\rm p}^2 = \frac{n_0 e^2}{m_e \epsilon_0} = \frac{8\pi^2 c\Delta t}{d}\frac{f_1^2 f_2^2}{(f_2 - f_1)(f_2 + f_1)} \tag{19}$$

Solving for n_0 gives $n_0 \approx 1.2$ cm^{-3}.

One thought on “M01E.2”

December 11, 2013 at 9:28 pm

Good solution.
There is a typo: in (5) and (8) there should be e not e^2 on the right-hand side.