1 May 2001, Electromagnetism, Problem 1

1.1 (a)

We need to solve Laplace's equation in the region between the plates:

$$\frac{1}{\rho}\frac{\partial\rho}{\partial}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Using the assumption that the electrical potential depends only on ϕ, we get:

$$\frac{\partial^2 V}{\partial\phi^2}=0$$
$$V=A\phi+B$$
$$V(0)=B=0$$
$$V(\phi_0)=A\phi_0=V_0$$
$$\boldsymbol{E}=-\nabla V=-\frac{1}{\rho}\frac{V_0}{\phi_0}\hat{\phi}$$

The assumption we used should be valid when ϕ_0 is small, say, $c >> (b+c)\phi_0$. The energy stored in the capacitor is:

$$U=\frac{\epsilon_0}{2}\int_0^d\int_b^{b+c}\int_0^{\phi_0}E^2\rho\,d\phi\,d\rho\,dz=\frac{\epsilon_0 d\phi_0}{2}\int_b^{b+c}\frac{1}{\rho}\frac{V_0^2}{\phi_0^2}\,d\rho=\frac{\epsilon_0 dV_0^2}{2\phi_0}\log\left(1+\frac{c}{b}\right) \tag{1}$$

We can also compute the capacitance by noting that:

$$U=\frac{1}{2}CV_0^2$$
$$C=\frac{\epsilon_0 d}{\phi_0}\log\left(1+\frac{c}{b}\right)$$

1.2 (b)

Now we have 10 capacitors connected in parallel, and capacitances in parallel add up, so:

$$C_T=10\frac{\epsilon_0 d}{\phi_0}\log\left(1+\frac{c}{b}\right)$$
$$10\phi_0=2\pi$$
$$C_T=50\frac{\epsilon_0 d}{\pi}\log\left(1+\frac{c}{b}\right) \tag{2}$$