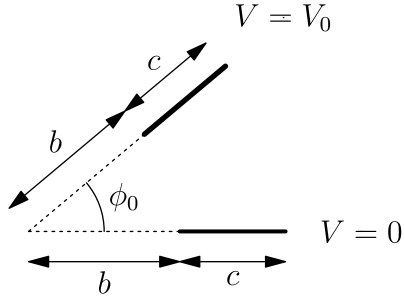
1 Dec 2021

M01E.1—Non-parallel Plate Capacitor

Problem

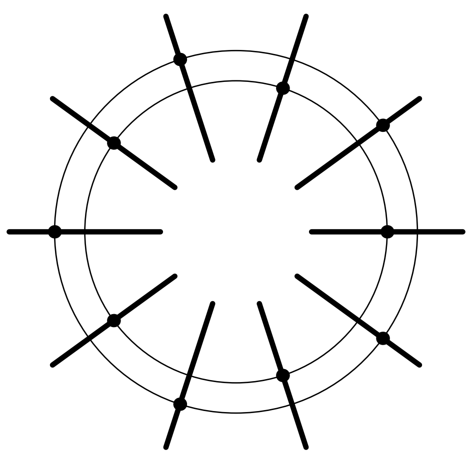
Two identical plates of length c and width d are separated by an angular separation of ϕ_0 as shown. THe plate at $\phi = 0$ is grounded, and the plate at $\phi = \phi_0$ is set at potential V_0.

lol



a) Compute the store energy in the capacitor. Assume that the electrical potential between the plates depends only on ϕ, and ignore fringe fields. (In which limit is this an allowed approximation?)

Now take ten in a cylindrical arrangement, and connect them as follows:



The odd plates are all connected together with a wire. The even plates are also all connected together. There is no direct connection between the odd and even plates. Assume a charge Q is placed on the even plates, and a charge $-Q$ on the odd plates.

b) Compute the total capacitance of this structure.

M01E.1

$Q = CV$; $U = \frac{1}{2}CV^2 = \frac{1}{2}QV$

a) $V(\phi) = f(\phi)$; $V(0) = 0$; $V(\phi_0) = V_0$

$$\frac{\partial^2 V}{\partial \phi^2} = 0 \rightarrow V(\phi) = s\phi + f$$

$V(0) = 0 \rightarrow f = 0$; $V(\phi_0) = V_0 \rightarrow s = \frac{V_0}{\phi_0}$

$$V(\phi) = \frac{V_0}{\phi_0}\phi$$

$$\vec{E} = -\nabla V(\phi) = -\frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} = -\frac{V_0}{\phi_0 r}\hat{\phi}$$

$$U = \int_{space} \frac{\varepsilon_0}{2}\vec{E}^2 d\tau = \frac{\varepsilon_0}{2}\int_0^{\phi_0}\int_b^{c+b}\int_0^{d} \frac{1}{r^2}\left(\frac{V_0}{\phi_0}\right)^2 r\, dz\, dr\, d\phi = \frac{\varepsilon_0}{2}\frac{V_0^2}{\phi_0^2}(\phi_0)(d)\int_b^{c+b}\frac{1}{r}dr$$
$$= \varepsilon_0 \frac{V_0^2 d}{2\phi_0}\left[\ln r\right]_b^{c+b} = \varepsilon_0 \frac{V_0^2 d}{2\phi_0}\ln\left(\frac{c+b}{b}\right)$$

$$\boxed{U = \varepsilon_0 \frac{V_0^2 d}{2\phi_0}\ln\left(1+\frac{c}{b}\right)}$$

Allowed limit: $c >> b$, $\phi_0 << 1$

$$U = \frac{1}{2}CV^2 \rightarrow C_1 = \frac{2U}{V^2} = \frac{2}{V_0^2}\varepsilon_0\frac{V_0^2 d}{2\phi_0}\ln\left(\frac{c+b}{b}\right) = \frac{\varepsilon_0 d}{\phi_0}\ln\left(\frac{c+b}{b}\right)$$

10 capacitors in parallel: $\boxed{C = 10C_1 = \frac{10\varepsilon_0 d}{\phi_0}\ln\left(\frac{c+b}{b}\right)}$