Seth Dorfman Prelims December 22, 2005

May 2000 QM

2) $(\vec{S}_e+\vec{S}_p)^2 = S_e^2 + 2\vec{S}_e\cdot\vec{S}_p + S_p^2 \qquad \vec{J} = \vec{S}_e+\vec{S}_p$

$\vec{S}_e\cdot\vec{S}_p = \frac{1}{2}\left[J^2 - S_e^2 - S_p^2\right]$

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Choose basis $|S_p^z S_e^z\rangle = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$

$|1,1\rangle = |\uparrow\uparrow\rangle \qquad J^2|1,1\rangle = 2\hbar^2|1,1\rangle$

$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \qquad J^2|1,0\rangle = 2\hbar^2|1,0\rangle$

$|1,-1\rangle = |\downarrow\downarrow\rangle \qquad J^2|1,-1\rangle = 2\hbar^2|1,-1\rangle$

$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad J^2|0,0\rangle = 0$

$\Rightarrow J^2|\uparrow\downarrow\rangle = J^2|\downarrow\uparrow\rangle$

$J^2|1,0\rangle = \frac{1}{\sqrt{2}}(2J^2|\uparrow\downarrow\rangle)$

$2\hbar^2\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)\right) = \frac{1}{\sqrt{2}}2J^2|\uparrow\downarrow\rangle$

$J^2|\uparrow\downarrow\rangle = \hbar^2(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle) = J^2|\downarrow\uparrow\rangle$

$J^2 = \hbar^2\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{pmatrix} \qquad S_p^2 = \hbar^2\begin{pmatrix}3/4&0&0&0\\0&3/4&0&0\\0&0&3/4&0\\0&0&0&3/4\end{pmatrix}$

$S_e^2 = \hbar^2\begin{pmatrix}3/4&0&0&0\\0&3/4&0&0\\0&0&3/4&0\\0&0&0&3/4\end{pmatrix}$

$\therefore \vec{S}_e\cdot\vec{S}_p = \frac{\hbar^2}{4}\begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$

Assume $\vec{B} = B\hat{z}$

$S_p^z = \frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \qquad S_e^z = \frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$

$H = 4\alpha\vec{S}_p\cdot\vec{S}_e + 2\beta S_p^z B + 2\gamma S_e^z B$

$$H = \hbar\begin{pmatrix}\alpha\hbar+(\beta+\gamma)B & 0 & 0 & 0\\ 0 & -\alpha\hbar+(\beta-\gamma)B & 2\alpha\hbar & 0\\ 0 & 2\alpha\hbar & -\alpha\hbar+(\gamma-\beta)B & 0\\ 0&0&0& \alpha\hbar-(\beta+\gamma)B\end{pmatrix}$$

$$\begin{vmatrix}\lambda^*+\alpha\hbar+(\gamma-\beta)B & -2\alpha\hbar\\ -2\alpha\hbar & \lambda^*+\alpha\hbar+(\gamma-\beta)B\end{vmatrix} = 0$$

$\lambda = \lambda^*\hbar$ (Factor of $\hbar$ dropped)

$(\lambda^*+\alpha\hbar)^2 - (\gamma-\beta)^2B^2 - 4\alpha^2\hbar^2 = 0$

$(\lambda^*+\alpha\hbar)^2 = (\beta-\gamma)^2B^2 + 4\alpha^2\hbar^2$

$$\lambda^* - \alpha\hbar = \pm\sqrt{(\beta-\gamma)^2 B^2 + 4\alpha^2\hbar^2}$$

$$\lambda^* = \alpha\hbar \pm\sqrt{(\beta-\gamma)^2 B^2 + 4\alpha^2\hbar^2}$$

$$\lambda = \hbar\lambda^* = \alpha\hbar^2 \pm \hbar\sqrt{4\alpha^2\hbar^2 + (\beta-\gamma)^2 B^2}$$

Thus the energy eigenvalues are:

$\alpha\hbar^2 + (\beta+\gamma)\hbar B$

$\alpha\hbar^2 + \hbar\sqrt{4\alpha^2\hbar^2 + (\beta-\gamma)^2 B^2}$

$\alpha\hbar^2 - \hbar\sqrt{4\alpha^2\hbar^2 + (\beta-\gamma)^2 B^2}$

$\alpha\hbar^2 - (\beta+\gamma)\hbar B$