Consider a spin-$\frac{1}{2}$ particle constrained to move on a 1D line with a harmonic oscillator potential and a magnetic field so that the Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z$$

The first energy level is not degenerate but all the other levels are doubly degenerate.

Now add a small magnetic field in the $\hat{x}$ direction with a magnitude proportional to x . The Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z + \alpha x S_x$$

Calculate the energy difference in the levels to lowest order.

Define the original the Hamiltonian:

$$H_0 = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z$$

And add the perturbation:

$$H' = \alpha x S_x$$

We define the basis of eigenstates of H_0 to be $|n_{k,} \pm\rangle$, where n_k is the quantum number for the kinetic energy, and $\pm$ defines either the + or - spin state. The energy from the state is $n\hbar\omega$.

Since we know:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & amp; 1 \\ 1 & amp; 0 \end{pmatrix}; \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & amp; 0 \\ 0 & amp; -1 \end{pmatrix}$$

so that the properties of S_x are:

$$S_x|+\rangle_z = \frac{\hbar}{2}|-\rangle_z; \qquad S_x|-\rangle_z = \frac{\hbar}{2}|+\rangle_z$$

We can also decompose the x operator:

$$\hat{x} = \frac{1}{\sqrt{2}}\left(a^\dagger + a\right)\sqrt{\frac{\hbar}{m\omega}}$$

where a and $a^\dagger$ are the annhiliation and creation operators. The energy perturbation to first order is given by:

$$E_n(\lambda) = E_n^0 + \sum_m \langle\varphi_m| H' |\varphi_m\rangle$$

Where we must diagonalize the states:

$$\begin{aligned} |\varphi_{m+}\rangle \quad amp; = \quad amp; \tfrac{1}{\sqrt{2}}\left(|m-1,+\rangle + |m,-\rangle\right) \\ |\varphi_{m-}\rangle \quad amp; = \quad amp; \tfrac{1}{\sqrt{2}}\left(|m-1,+\rangle - |m,-\rangle\right) \end{aligned}$$

So the change in energy is given by:

$$\Delta E_m(\lambda) = \langle\varphi_m| H' |\varphi_m\rangle$$

Now it simply remains to calculate the values of $\langle\varphi_{m\pm}| H' |\varphi_{m\pm}\rangle$:

$$\langle\varphi_{m\pm}| H' |\varphi_{m\pm}\rangle = \langle\varphi_{m\pm}|\alpha\frac{1}{\sqrt{2}}\left(a^\dagger + a\right)\sqrt{\frac{\hbar}{m\omega}}S_x|\varphi_{m\pm}\rangle$$

So that for $|\varphi_{m+}\rangle$:

$$\frac{1}{\sqrt{2}}\left(\langle m-1,+| + \langle m,-|\right)\alpha\frac{1}{\sqrt{2}}\left(a^\dagger + a\right)\sqrt{\frac{\hbar}{m\omega}}S_x\frac{1}{\sqrt{2}}\left(|m-1,+\rangle + |m,-\rangle\right) = \alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{\hbar}{2}\sqrt{n}$$

And for $\left|\varphi_{m-}\right\rangle$:

$$\frac{1}{\sqrt{2}}\left(\langle m-1,+|-\langle m,-|\right)\alpha\frac{1}{\sqrt{2}}\left(a^\dagger+a\right)\sqrt{\frac{\hbar}{m\omega}}S_x\frac{1}{\sqrt{2}}\left(|m-1,+\rangle-|m,-\rangle\right)=-\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{\hbar}{2}\sqrt{n}$$

So that the total energy difference is just given by:

$$\Delta E\left(|\varphi_{m+}\rangle\right)=\sqrt{\frac{\hbar^3\alpha^2 n}{8m\omega}};\qquad \Delta E\left(|\varphi_{m-}\rangle\right)=-\sqrt{\frac{\hbar^3\alpha^2 n}{8m\omega}}$$

We must treat the ground state separately, since it is nondegenerate. We must take it out to second order:

$$\Delta E\left(|0\rangle\right)=\sum_{p\neq 0}\frac{\left|\left\langle\varphi_p^i\right|H'\left|0\right\rangle\right|^2}{-E_p^0}$$

Finding the nonzero values of this, we get:

$$\langle 1,+|\,H'\,|0,-\rangle=\langle 1,-|\,\alpha\frac{1}{\sqrt{2}}\left(a^\dagger+a\right)\sqrt{\frac{\hbar}{m\omega}}S_x\,|0,-\rangle=\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{\hbar}{2}$$

as the only one. Thus:

$$\Delta E\left(|0\rangle\right)=\frac{\left|\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{\hbar}{2}\right|^2}{-\hbar\omega}=-\alpha^2\frac{\hbar^2}{8m\omega^2}$$

At last we get:

$$\begin{matrix}\Delta E_n & amp;= & amp;-\alpha^2\frac{\hbar^2}{8m\omega^2}; & n=0 \\ & amp;= & amp;\begin{cases}\alpha\sqrt{\frac{\hbar^3 n}{8m\omega}} \\ -\alpha\sqrt{\frac{\hbar^3 n}{8m\omega}}\end{cases} & nbsp;; \quad n \quad gt;0\end{matrix}$$

where the pair for $n>0$ indicates the splitting of a degenerate pair.