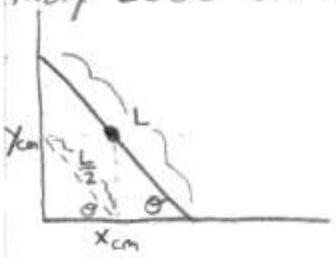
May 2000 CM #2 (Revised)



$I_{cm} = \int \rho r^2 \, dr$

$= 2\rho \int_0^{L/2} r^2 \, dr$

$= 2\frac{m}{L}\frac{(L/2)^3}{3}$

$I_{cm} = \frac{1}{12} m L^2$

$x_{cm} = \frac{L}{2}\cos\theta$

$y_{cm} = \frac{L}{2}\sin\theta$

$v_{cm}^2 = \dot{x}_{cm}^2 + \dot{y}_{cm}^2$

$v_{cm}^2 = \left(-\frac{L}{2}\sin\theta\dot{\theta}\right)^2 + \left(\frac{L}{2}\cos\theta\dot{\theta}\right)^2$

$v_{cm}^2 = \left(\frac{L}{2}\dot{\theta}\right)^2$

$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \dot{\theta}^2$

$T = \frac{1}{2} m \frac{L^2}{4}\dot{\theta}^2 + \frac{1}{2}\frac{1}{12} m L^2 \dot{\theta}^2$

$T = \frac{1}{2} m L^2 \dot{\theta}^2 \left(\frac{1}{4} + \frac{1}{12}\right)$

$T = \frac{1}{2} m L^2 \dot{\theta}^2 \left(\frac{3+1}{12}\right)$

$T = \frac{1}{6} m L^2 \dot{\theta}^2$

$U = m g y_{cm}$

$U = m g \frac{L}{2}\sin\theta$

$E = \frac{1}{6} m L^2 \dot{\theta}^2 + m g \frac{L}{2}\sin\theta$

Ladder released from rest at $\theta = \alpha \Rightarrow E = m g \frac{L}{2}\sin\alpha$

$m g \frac{L}{2}\sin\alpha = \frac{1}{6} m L^2 \dot{\theta}^2 + m g \frac{L}{2}\sin\theta$

$g\sin\alpha = \frac{1}{3} L \dot{\theta}^2 + g\sin\theta$

$\dot{\theta}^2 = \frac{3g}{L}(\sin\alpha - \sin\theta)$

$2\dot{\theta}\ddot{\theta} = -\frac{3g}{L}\cos\theta\dot{\theta}$

$\ddot{\theta} = -\frac{3g}{2L}\cos\theta$

$x_{cm} = \frac{L}{2}\cos\theta$

$\dot{x}_{cm} = -\frac{L}{2}\sin\theta\dot{\theta}$

$\ddot{x}_{cm} = -\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta}$

$\ddot{x}_{cm} = \frac{L}{2}\left[-\frac{3g}{L}(\sin\alpha - \sin\theta)\cos\theta + \frac{3g}{2L}\cos\theta\sin\theta\right]$

$\ddot{x}_{cm} = \frac{3g}{4}\left[-2(\sin\alpha - \sin\theta)\cos\theta + \cos\theta\sin\theta\right]$

$\ddot{x}_{cm} = \frac{3g}{4}\cos\theta\left[3\sin\theta - 2\sin\alpha\right]$

The wall can push, but cannot pull on the ladder. Thus if our expression for $\ddot{x}_{cm} < 0$ the result is unphysical and the assuption used to derive it (ladder not seperated from wall) no longer holds. Thus the ladder seperates from the wall when:

$3\sin\theta = 2\sin\alpha$

$\sin\theta = \frac{2}{3}\sin\alpha$

$\theta = \text{Arc}\sin\left[\frac{2}{3}\sin\alpha\right]$