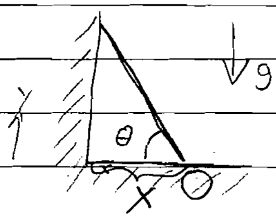
May 2000 # 2 (CM)

Uniform ladder



initial angle α

what angle does it separate from the wall?

Kinetic energy T

$T=\sum_\alpha \frac{1}{2} m_\alpha V_\alpha^2$

Body coordinate system: origin at the bottom of the ladder: moving at $\vec{V}$

$\vec{V}_\alpha = \vec{V} + \vec{\omega}\times\vec{r}_\alpha$ where $\vec{\omega} = \dot{\vec{\theta}}$, $\vec{r}_\alpha$ is the position vector from the body origin to the particle

$V_\alpha^2 = V^2 + 2\vec{V}\cdot\vec{\omega}\times\vec{r}_\alpha + (\vec{\omega}\times\vec{r}_\alpha)^2$

$T = \sum_\alpha \frac{1}{2} m_\alpha V^2 + \sum_\alpha m_\alpha \vec{V}\cdot\vec{\omega}\times\vec{r}_\alpha + \sum_\alpha \frac{1}{2} m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$

$T = \frac{1}{2} M V^2 + \vec{V}\cdot\vec{\omega}\times M\vec{R} + \frac{1}{2} I_{ij}\omega_i\omega_j$

$\vec{R}$ is the center of mass vector in the body frame.

Since $\omega = \omega_z$, $\frac{1}{2} I_{ij}\omega_i\omega_j = \frac{1}{2} I\omega^2$

Uniform Rod pivoting at end: $I = \frac{1}{3} M L^2$

if body coord system at CoM, $\sum m_\alpha \vec{r}_\alpha = M\vec{R} = 0$

$V^2 = \dot{x}^2 + \dot{y}^2$

$\Rightarrow$ $L \to 2L$

$T = \frac{1}{2} M L^2 \dot{\theta}^2 + \frac{1}{6} M L^2 \dot{\theta}^2 = \frac{2}{3} M L^2 \dot{\theta}^2$

$\Rightarrow$ $\sin\theta_f = \frac{2}{3}\sin\alpha$

Here, $\vec{R} = -\frac{L}{2}\cos\theta\,\hat{x} + \frac{L}{2}\sin\theta\,\hat{y}$

$\vec{\omega}\times\vec{R} = \dot{\theta}\left(\frac{L}{2}\cos\theta\,\hat{y} - \frac{L}{2}\sin\theta\right)\hat{x}$ $\quad \boxed{\vec{\omega} = -\dot{\theta}\,\hat{z}}$

$\vec{V} = \dot{x}\,\hat{x}$ $\quad \vec{V}\cdot\vec{\omega}\times\vec{R} = +\frac{1}{2} L\dot{x}\dot{\theta}\sin\theta = +\frac{1}{2} L\dot{x}\dot{\theta}\sin\theta$

$\Rightarrow T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} M L\dot{x}\dot{\theta}\sin\theta + \frac{1}{6} M L^2\dot{\theta}^2$

$U = \frac{1}{2} M g L\sin\theta$

$x = L\cos\theta$ while in contact with the wall

$\dot{x} = -L\dot{\theta}\sin\theta$

$T = \frac{1}{6} M L^2 \dot{\theta}^2$

initially $T = 0$

conservation of energy: $\frac{1}{6}\dot{\theta}^2 M L^2 + \frac{1}{2} M g L\sin\theta = \frac{1}{2} M g L\sin\alpha$

$\dot{\theta}^2 = \frac{3g}{L}\left(\sin\alpha - \sin\theta\right)$

The x coordinate of the center of mass:

$x_{cm} = \frac{L}{2}\cos\theta$

The only acceleration/force on the ladder in the x direction is the normal force from the wall, so the force goes to 0 when $\ddot{x}_{cm} = 0$

$\dot{x}_{cm} = -\frac{L}{2}\dot{\theta}\sin\theta$

$\ddot{x}_{cm} = -\frac{L}{2}(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta)$

$N \to 0$ when $\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta = 0$ (Gives separation angle θ_{crit})

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{L}\frac{d}{d\theta}\left(\sin\alpha - \sin\theta\right)\dot{\theta}$$

$$\ddot{\theta} = -\frac{3g}{2L}\left(\cos\theta\right)$$

$$\frac{3g}{L}(\sin\alpha - \sin\theta)\cos\theta - \frac{3g}{2L}\sin\theta\cos\theta = 0$$

$$\sin\alpha - \sin\theta - \frac{1}{2}\sin\theta = 0$$

$$\sin\alpha = \frac{3}{2}\sin\theta$$

$$\boxed{\sin\theta_c = \frac{2}{3}\sin\alpha}$$

Ignore incorrect solution

let $x = \sin\theta$, $B = \frac{1}{3}\sin\alpha$

$$x^3 + \frac{1}{2}x = B$$

Find s, t st. $3st = \frac{1}{2}$, $s^3 - t^3 = B$. Then $y = s - t$ is a solution

$$\left(\frac{1}{6t}\right)^3 - t^3 = B \Rightarrow t^6 + Bt^3 - \frac{1}{216} = 0$$

$$\Rightarrow t^3 = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} + \frac{1}{216}}$$ take the positive root

$$t^3 = -\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{1}{216}} \qquad s^3 = B + t^3 = \frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{1}{216}}$$

$$y = s - t = \sqrt[3]{\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{1}{216}}} - \sqrt[3]{-\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{1}{216}}}$$

or $$\sin\theta_{crit} = \sqrt[3]{\frac{\sin\alpha}{6} + \sqrt{\frac{\sin^2\alpha}{36} + \frac{1}{216}}} - \sqrt[3]{\frac{-\sin\alpha}{6} + \sqrt{\frac{\sin^2\alpha}{36} + \frac{1}{216}}}$$