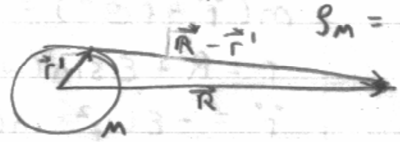
Seth Dorfman Prelims December 21, 2005

May 2000 CM

$\rho_M = \frac{M}{2\pi a}$



1) a. $\phi_M = -G \int \frac{\rho_M}{|\vec{R}-\vec{r}'|} \, dr'$

$$\frac{1}{|\vec{R}-\vec{r}'|} = \frac{1}{\sqrt{R^2 - 2\vec{R}\cdot\vec{r}' + a^2}}$$
$$= \frac{1}{R}\frac{1}{\sqrt{1 + \frac{1}{R^2}(a^2 - 2Ra\cos\theta)}}$$
$$\approx \frac{1}{R}\left(1 - \frac{1}{2R^2}(a^2 - 2Ra\cos\theta) + \frac{1}{8}\frac{1}{R^4}(a^2 - 2Ra\cos\theta)^2\right)$$
$$\approx \frac{1}{R}\left(1 + \frac{1}{R}a\cos\theta - \frac{1}{2}\frac{a^2}{R^2} + \frac{1}{2}\frac{a^2}{R^2}\cos^2\theta\right)$$

$$\phi_M \approx -G\rho_M \int_0^{2\pi} \frac{1}{R}\left(1 + \frac{1}{R}a\cos\theta - \frac{1}{2}\frac{a^2}{R^2}\sin^2\theta\right) a\,d\theta$$
$$= -G\rho_M \frac{a}{R}\int_0^{2\pi} 1 + \frac{a}{R}\cos\theta - \frac{1}{2}\left(\frac{a}{R}\right)^2 \sin^2\theta \, d\theta$$
$$\approx -G\rho_M \frac{a}{R}\left(2\pi + 0 - \frac{\pi}{2}\left(\frac{a}{R}\right)^2\right)$$
$$= \frac{-GM}{2\pi a}\frac{a}{R}\left(2\pi - \frac{\pi}{2}\left(\frac{a}{R}\right)^2\right)$$
$$\phi_M = \frac{-GM}{R}\left(1 - \frac{1}{4}\left(\frac{a}{R}\right)^2\right)$$

b. Consider ~~a~~ the circular orbit of a particle of mass m a distance R away.

$$ma = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial R} m\phi_M$$
$$ma = m\frac{\partial}{\partial R}\left[\frac{GM}{R}\left(1 - \frac{1}{4}\left(\frac{a}{R}\right)^2\right)\right]$$
$$ma = -\frac{GMm}{R^2} + \frac{1}{4}GMma^2 \cdot \frac{3}{R^4}$$
$$ma = -\frac{GMm}{R^2}\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right)$$

$\vec{a} = \frac{\partial^2}{\partial t^2}(r\hat{r}) = \frac{\partial}{\partial t}(r\dot{\theta}\hat{\theta}) = -r\dot{\theta}^2\hat{r} = -r\omega^2\hat{r}$ for uniform circular motion

$$-mR\omega^2 = \frac{-GMm}{R^2}\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right)$$
$$\omega^2 = \frac{GM}{R^3}\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right)$$
$$\omega = \sqrt{\frac{GM}{R^3}}\sqrt{1 - \frac{3}{4}\left(\frac{a}{R}\right)^2}$$
$$\omega \approx \frac{1}{R}\sqrt{\frac{GM}{R}}\left(1 - \frac{3}{8}\left(\frac{a}{R}\right)^2\right)$$

c. Now $\vec{a} = \frac{\partial^2}{\partial t^2}(r\hat{r})$
$$= \frac{\partial}{\partial t}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$
$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$r = R + \varepsilon\sin(\omega_r t)$

No forces in $\hat{\theta}$ direction $\Rightarrow$ $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\frac{d}{dt}\left(\dot{\theta}\left(R^2 + 2R\varepsilon\sin(\omega_r t) + \underbrace{\varepsilon^2\sin^2(\omega_r t)}_{\text{small}}\right)\right) = 0$$

$$\Rightarrow \dot{\theta} = \omega_0\left(1 - \frac{2\varepsilon}{R}\sin(\omega_r t)\right)$$

In the $\hat{r}$ direction:

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-GMm}{r^2}\left(1 - \frac{3}{4}\left(\frac{a}{r}\right)^2\right)$$

$$r = R + \varepsilon\sin(w_r t) \qquad \dot{\theta} = w_0\left(1 - \frac{2\varepsilon}{R}\sin(w_r t)\right)$$

$$\ddot{r} = -\varepsilon w_r^2 \sin(w_r t)$$

$$r\dot{\theta}^2 = R w_0^2 - R w_0^2 \cdot \frac{4\varepsilon}{R}\sin(w_r t) + w_0^2 \varepsilon \sin(w_r t)$$

$$\frac{1}{r^2} = \frac{1}{(R + \varepsilon\sin(w_r t))^2} = \frac{1}{R^2}\frac{1}{(1 + \varepsilon\sin(w_r t)/R)^2} \approx \frac{1}{R^2}\left(1 - 2\frac{\varepsilon}{R}\sin(w_r t)\right)$$

The equation becomes:

$$-\varepsilon w_r^2 \sin(w_r t) - R w_0^2 + 3\varepsilon w_0^2 \sin(w_r t) = \frac{-GM}{R^2}\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right)$$
$$+ GM\frac{2\varepsilon}{R^3}\sin(w_r t)\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right) - \frac{GM}{R^2}\left(\frac{3}{4}\left(\frac{a}{R}\right)^2 2\frac{\varepsilon}{R}\sin(w_r t)\right)$$

Subtracting the 0^{th} order equation and canceling the $\varepsilon\sin(w_r t)$ factors:

$$-w_r^2 + 3w_0^2 = -GM\left(\frac{-2}{R^3} + \frac{3}{R^3}\left(\frac{a}{R}\right)^2\right)$$

$$w_r^2 = \frac{GM}{R^3}\left(-2 + 3\left(\frac{a}{R}\right)^2\right) + 3w_0^2$$

$$w_r^2 = \frac{GM}{R^3}\left[-2 + 3\left(\frac{a}{R}\right)^2 + 3\left(1 - \frac{3}{4}\left(\frac{a}{R}\right)^2\right)\right]$$

$$w_r^2 = \frac{GM}{R^3}\left[1 + \frac{3}{4}\left(\frac{a}{R}\right)^2\right]$$

$$w_r = \frac{1}{R}\sqrt{\frac{GM}{R}}\sqrt{1 + \frac{3}{4}\left(\frac{a}{R}\right)^2}$$

$$w_r \approx \frac{1}{R}\sqrt{\frac{GM}{R}}\left(1 + \frac{3}{8}\left(\frac{a}{R}\right)^2\right)$$

$$w_r - w_0 = \frac{1}{R}\sqrt{\frac{GM}{R}}\left(\frac{3}{4}\left(\frac{a}{R}\right)^2\right)$$

$$\frac{w_r - w_0}{w_0} = \frac{3}{4}\left(\frac{a}{R}\right)^2 \Big/ \left[1 - \frac{3}{8}\left(\frac{a}{R}\right)^2\right]$$

$$\frac{w_r - w_0}{w_0} = 2 \cdot \frac{1}{\frac{8}{3}\left(\frac{R}{a}\right)^2 - 1}$$

$$\Delta\phi = \frac{2\pi(w_r - w_0)}{w_0} = \frac{4\pi}{\frac{8}{3}\left(\frac{R}{a}\right)^2 - 1}$$

$$\Delta\phi \approx \frac{3}{2}\pi\left(\frac{a}{R}\right)^2$$