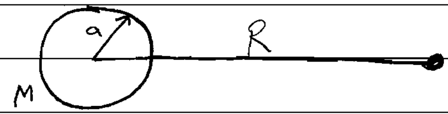
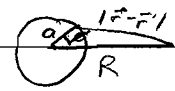
May 2000 #1 (CM)



$R >> a$

a) $\Phi = -G \int \frac{dM}{|\vec{r}-\vec{r}'|}$ $\qquad dM = \lambda dl = \lambda a d\phi$ $\qquad \lambda = \frac{M}{2\pi a}$

$$\Phi = -G\lambda a \int_0^{2\pi} \frac{d\phi}{|\vec{r}-\vec{r}'|}$$



$|\vec{r}-\vec{r}'| = \sqrt{R^2+a^2-2Ra\cos\phi}$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R}\cdot\left(1+\frac{a^2}{R^2}-\frac{2a}{R}\cos\phi\right)^{-1/2}$$

$$= \frac{1}{R}\left[1-\frac{1}{2}\left(\frac{a^2}{R^2}-\frac{2a}{R}\cos\phi\right)+\frac{3}{8}\left(\frac{a^2}{R^2}-\frac{2a}{R}\cos\phi\right)^2+\dots\right]$$

$$= \frac{1}{R}\left[1+\frac{a}{R}\cos\phi+\left(\frac{a}{R}\right)^2\frac{3\cos^2\phi-1}{2}+\dots\right]$$

$$\Phi = \frac{-G\lambda a}{R}\int_0^{2\pi} d\phi \left(\underset{2\pi}{\underset{\downarrow}{1}} + \underset{0}{\underset{\downarrow}{\frac{a}{R}\cos\phi}} + \frac{a^2}{R^2}\underset{\frac{\pi}{2}}{\underset{\downarrow}{\frac{3\cos^2\phi-1}{2}}}\right)$$

$$\Phi = \frac{-G\lambda a}{R}\left(2\pi+\frac{\pi}{2}\frac{a^2}{R^2}\right) = \frac{-GM}{R}\left(1+\frac{1}{4}\frac{a^2}{R^2}\right)$$

$$U = m\Phi = \frac{-GmM}{R}\left(1+\frac{1}{4}\frac{a^2}{R^2}\right)$$

$$L = T-U = \frac{1}{2}m(\dot{R}^2+R^2\dot{\phi}^2)+\frac{GmM}{R}\left(1+\frac{1}{4}\frac{a^2}{R^2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi} \qquad \frac{d}{dt}\left(mR^2\dot{\phi}\right) = 0 \qquad \dot{\phi} \equiv \frac{l}{mR^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{R}}\right) = \frac{\partial L}{\partial R} \qquad m\ddot{R} = mR\dot{\phi}^2 - \frac{GmM}{R^2}\left(1+\frac{3}{4}\frac{a^2}{R^2}\right)$$

circular orbit: $\ddot{R}=0 \Rightarrow \dot{\phi}^2 = \omega_0^2 = \frac{GM}{R^3}\left(1+\frac{3}{4}\frac{a^2}{R^2}\right)$

$$\omega_0 \approx \left(\frac{GM}{R_0^3}\right)^{1/2}\left(1+\frac{3}{8}\frac{a^2}{R^2}\right)$$

c. small radial perturbation: $R \to R_0 + \epsilon$ $\ddot{R} = \ddot{\epsilon}$

$$\ddot{\epsilon} = R\dot{\phi}^2 - \frac{GM}{R^2}\left(1 + \frac{3}{4}\frac{a^2}{R^2}\right)$$

expand $\frac{GM}{R^2}\left(1 + \frac{3}{4}\frac{a^2}{R^2}\right)$ about R_0:

$$GM\left[\frac{2}{R^3} - \frac{3a^2}{R^5}\right] \to \frac{GM}{R_0^2}\left(1 + \frac{3}{4}\frac{a^2}{R_0^2}\right) - \frac{GM}{R_0^3}\left(2 + \frac{3a^2}{R_0^2}\right)\epsilon$$

$$R\dot{\phi}^2 = \frac{l^2}{m^2R^3} \quad \text{expand about } R_0: \quad \frac{l^2}{m^2(R_0+\epsilon)^3} = \frac{l^2}{m^2R_0^3\left(1+\frac{\epsilon}{R_0}\right)^3} \simeq \frac{l^2}{m^2R_0^3}\left(1 - \frac{3\epsilon}{R_0}\right)$$

· because l is a constant, while $\dot{\phi}$ is not, as R changes

$$\ddot{\epsilon} = \frac{l^2}{m^2R_0^3} - \frac{3l^2}{m^2R_0^3}\frac{\epsilon}{R_0} - \frac{GM}{R_0^2}\left(1 + \frac{3}{4}\frac{a^2}{R_0^2}\right) + \frac{GM}{R_0^3}\left(2 + \frac{3a^2}{R_0^2}\right)\epsilon$$

$$\frac{l^2}{m^2R_0^3} = R_0\dot{\phi}_0^2 = \frac{GM}{R_0^2}\left(1 + \frac{3}{4}\frac{a^2}{R_0^2}\right)$$

$$\ddot{\epsilon} = -\frac{3GM}{R_0^3}\left(1 + \frac{3}{4}\frac{a^2}{R_0^2}\right)\epsilon + \frac{2GM}{R_0^3}\epsilon + \frac{3GM}{R_0^3}\frac{a^2}{R_0^2}\epsilon$$

$$\ddot{\epsilon} = \left(-\frac{GM}{R_0^3} + \frac{3}{4}\frac{GMa^2}{R_0^5}\right)\epsilon = -\frac{GM}{R_0^3}\left(1 - \frac{3a^2}{4R_0^2}\right)\epsilon$$

$$\omega_r^2 = \frac{GM}{R_0^3}\left(1 - \frac{3}{4}\frac{a^2}{R_0^2}\right)$$

$$\omega_r \approx \sqrt{\frac{GM}{R_0^3}}\left(1 - \frac{3}{8}\frac{a^2}{R_0^2}\right)$$

$$\frac{\omega_r - \omega_0}{\omega_0} \simeq \frac{\left(\frac{GM}{R_0^3}\right)^{1/2}\left(-\frac{3}{8}\frac{a^2}{R_0^2}\right)\cdot 2}{\left(\frac{GM}{R_0^3}\right)^{1/2}} = -\frac{3}{4}\frac{a^2}{R_0^2}$$

$$\Delta\phi = 2\pi\left(\frac{\omega_r - \omega_0}{\omega_0}\right) = -\frac{3\pi}{2}\frac{a^2}{R_0^2}$$