We consider nested toroidal solenoids:

There are $N_1$ turns in the outer solenoid, and $N_2$ turns in the inner. The inner has resistance $R$, the outer has negligible resistance. We are asked to find (a) the power dissipation given AC voltage, and (b) the relevant inductances.

(a) To start we can draw a circuit diagram:

We can write down the voltage about each loop

\[ V_1 = L_1 \dot{I}_1 + M \dot{I}_2 \]

\[ M \dot{I}_1 = L_2 \dot{I}_2 + I_2 R \]

Since these ideal solenoids do not leak flux, the only power dissipated is from resistance

\[ P = IV = I_2^2 R \]

Eliminating $I_1$ we can solve a first order differential equation for $I_2$

\[ \frac{V}{L_1} = \left( \frac{L_1 L_2}{M} + M \right) \dot{I}_2 + \frac{L_1}{M} RI_2 \]

Now setting $V = V_0 e^{i\omega t}$ we see

\[ \frac{M^2 + L_1 L_2}{L_1 R} \dot{I}_2 + I_2 = \frac{V_0}{R} \frac{M}{L_1} e^{i\omega t} \]

But the differential equation

\[ \alpha \dot{x} + x = \beta e^{i\omega t} \]
can be solved with a transient solution 

\[ x(t) = x_0 e^{-\beta t/\alpha} \]

plus a particular solution \( Ae^{i\omega t} \) where \(^1\)

\[ A(1 + i\omega \alpha) = \beta \]

One can solve the complex amplitude

\[ A = \frac{\beta}{\sqrt{1 + (\alpha \omega)^2}} e^{-i \tan^{-1}(\alpha \omega)} \]

so the power dissipated is

\[ P = I_2^2 R = A^2 R \langle \sin^2(\omega t) \rangle = \frac{\beta^2 R}{1 + (\alpha \omega)^2} \]

or adding back physical parameters

\[ P = \frac{V_0^2 M^2 R/2}{(RL_1)^2 + \omega^2 (M^2 + L_1 L_2)^2} \]

\[ = \frac{V_0^2 /2R}{(L_1/M)^2 + \omega^2 (M/R)^2 (1 + L_1 L_2 /M^2)^2} \]

Let us remark that

- \( P \to 0 \) as \( R \to \infty \) since the coupled current \( I_2 \to 0 \) for large \( R \).
- \( P \to 0 \) as \( \omega \to \infty \) since the reactance \( Z = i \omega L \) overshadows \( R \) at high frequency.

(b) Recall that inductance is defined

\[ \Phi = LI \]

We can calculate

\[ \Phi = N \int B \cdot dA = \frac{\mu_0 N^2 I}{2\pi} l \int \frac{dr}{r} \]

where we used Ampere’s Law

\[ \int B \cdot dl = \mu_0 I_{enc} \]

to find

\[ B(r) = \frac{\mu_0 N I}{2\pi r} \]

and defined \( l, w \) from the flux cross-section.

It follows that

\[ L_1 = \frac{\mu_0 N_1^2}{2\pi} 3s \ln \left( 1 + \frac{3s}{a} \right) \]

similarly

\[ L_2 = \frac{\mu_0 N_2^2}{2\pi} s \ln \left( 1 + \frac{s}{a + s} \right) \]

and

\[ M = \frac{\mu_0 N_1 N_2}{2\pi} s \ln \left( 1 + \frac{s}{a + s} \right) \]

In the limit \( s \ll a \), we see \( L_1 L_2 \approx M^2 \).

\(^1\) just plug in \( Ae^{i\omega t} \)