Seth Dorfman Prelims December 22, 2005

May 2000 EM

3) a. Let L_1 = self inductance of outer solenoid

M_{12} = mutual inductance of solenoids

L_2 = self inductance of inner solenoid

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad V_1 = V_0 e^{i\omega t}, \quad V_2 = 0$$

$$V_1 = L_1 \frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt} \qquad 0 = M_{12}\frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

$$V_1 = i\omega(L_1 I_1 + M_{12} I_2) \qquad 0 = i\omega(M_{12} I_1 + L_2 I_2)$$

$$V_1 = i\omega\left(L_1 I_1 - \frac{M_{12}^2}{L_2} I_1\right) \qquad I_2 = \frac{-M_{12}}{L_2} I_1$$

$$V_1 = I_1 i\omega\left(L_1 - \frac{M_{12}^2}{L_2}\right)$$

$$I_1 = V_1 \frac{i}{\omega} \frac{1}{M_{12}^2/L_2 - L_1}$$

$$I_1 = \mathrm{Re}\left[V_0 e^{i\omega t} \frac{i}{\omega} \frac{1}{M_{12}^2/L_2 - L_1}\right]$$

$$I_1 = V_0 \sin(\omega t) \frac{1}{\omega} \frac{1}{L_1 - M_{12}^2/L_2}$$

$$P = |V_1 I_1 + \overset{0}{V_2} I_2|$$

$$P = \left|V_0 \cos(\omega t) \cdot V_0 \sin(\omega t) \cdot \frac{1}{\omega} \frac{1}{L_1 - M_{12}^2/L_2}\right|$$

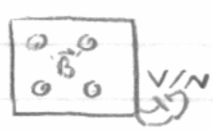
$$P = \frac{1}{2\omega} V_0^2 |\sin(2\omega t)| \frac{1}{L_1 - M_{12}^2/L_2}$$

$$P = \frac{1}{2\omega} \frac{L_2}{L_1 L_2 - M_{12}^2} V_0^2 |\sin(2\omega t)|$$

$$\langle|\sin(2\omega t)|\rangle = \frac{1}{2\omega} \cdot 2 \frac{1}{T/2} = \frac{2}{\omega} \cdot \frac{1}{T} = \frac{2}{\omega}\frac{2\omega}{2\pi} = \frac{2}{\pi}$$

$$\Rightarrow \langle P \rangle = \frac{1}{\pi\omega} \frac{L_2}{L_1 L_2 - M_{12}^2} V_0^2$$

b. First, consider a single solenoid of inner diameter b and side d. For a single loop:



$$\mathcal{E} = \frac{V}{N} \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$$

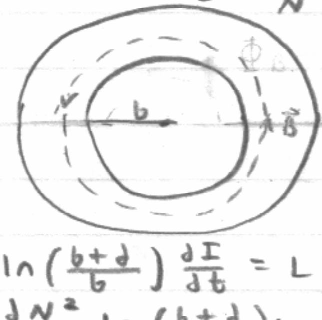
$$\mathcal{E} = \frac{-V}{N} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} I_{enc}$$

$$2\pi r B = \frac{4\pi}{c} I N$$

$$B = \frac{2IN}{cr}$$



$$\Phi_B = \int_b^{b+d}\int_0^d B(r)\,dz\,dr$$

$$= d\int_b^{b+d} \frac{2IN}{cr}\,dr$$

$$\Phi_B = d\,\frac{2IN}{c}\ln\left(\frac{b+d}{b}\right)$$

$$\frac{\partial \Phi_B}{\partial t} = \frac{2dN}{c}\ln\left(\frac{b+d}{b}\right)\frac{dI}{dt}$$

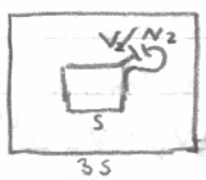
$$V = \frac{N}{c}\frac{\partial \Phi_B}{\partial t} = \frac{2dN^2}{c^2}\ln\left(\frac{b+d}{b}\right)\frac{dI}{dt} = L\frac{dI}{dt}$$

$$\Rightarrow L = \frac{2dN^2}{c^2}\ln\left(\frac{b+d}{b}\right)$$

$L_1 = L(b = a,\ d = 3s)$ $\qquad$ $L_2 = L(b = a+s,\ d = s)$

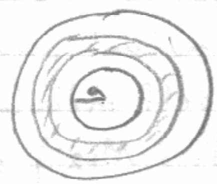
$L_1 = \frac{6 s N_1^2}{c^2} \ln\left(\frac{a+3s}{a}\right)$ $\qquad$ $L_2 = \frac{2 s N_2^2}{c^2} \ln\left(\frac{a+2s}{a+s}\right)$

Calculating M_{12}:



$\mathcal{E} = \frac{-V_2}{N_2}$

voltage on inner conductor



$B = \frac{2 I_1 N_1}{c r}$

induced by current in outer conductor

through the inner conductor:

$$\Phi_B = \int_0^s \int_{a+s}^{a+2s} B(r)\, dr\, dz$$

$$= s \int_{a+s}^{a+2s} \frac{2 I_1 N_1}{c r}\, dr$$

$$\Phi_B = s\, \frac{2 I_1 N_1}{c} \ln\left(\frac{a+2s}{a+s}\right)$$

$$\frac{d\Phi_B}{dt} = \frac{2 N_1 s}{c} \ln\left(\frac{a+2s}{a+s}\right) \frac{dI_1}{dt}$$

$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi_B}{dt}$$

$$V_2 = \frac{N_2}{c} \frac{d\Phi_B}{dt}$$

$$V_2 = \frac{2 N_1 N_2 s}{c^2} \ln\left(\frac{a+2s}{a+s}\right) \frac{dI_1}{dt} = M_{12} \frac{dI_1}{dt}$$

$$\Rightarrow M_{12} = \frac{2 s N_1 N_2}{c^2} \ln\left(\frac{a+2s}{a+s}\right)$$