M00-2

Dielectric & Image Charge

a) with an infinite conductor we place an image charge $-q$ at $-z$ for all time

$$F = -\frac{q^2}{(2z_0)^2} = -\frac{q^2}{4z^2} = m\ddot{z}$$

so $$\ddot{z} = -\frac{q^2}{4m} z^{-2}$$

$$\dot{z}\ddot{z} = -\frac{q^2}{4m}\dot{z} z^{-2}$$

$$\frac{1}{2}\frac{d}{dt}(\dot{z}^2) = \frac{q^2}{4m}\frac{d}{dt}(z^{-1})$$

$$d(\dot{z}^2) = \frac{q^2}{2m} d(z^{-1})$$

$$\dot{z}^2 = \frac{q^2}{2m} z^{-1} + C$$

we know $\dot{z} = 0$ when $z = z_0$

so $$\dot{z}^2 = \frac{q^2}{2mz} - \frac{q^2}{2mz_0}$$

$$\frac{dz}{dt} = \sqrt{\frac{q^2}{2mz} - \frac{q^2}{2mz_0}}$$

$$dt = \frac{dz}{\sqrt{\frac{q^2}{2mz} - \frac{q^2}{2mz_0}}} = \sqrt{\frac{2mz_0}{q^2}} \frac{dz}{\sqrt{\frac{z_0}{z} - 1}}$$

$$\int_0^t dt = \sqrt{\frac{2mz_0}{q^2}} \int_{z_0}^0 \frac{dz}{\sqrt{\frac{z_0}{z} - 1}}$$

let $x = \frac{z_0}{z}$ $\quad dx = -\frac{z_0}{z^2} dz$

$z = \frac{z_0}{x}$ $\quad dz = -\frac{z_0}{x^2} dx$

so $$\boxed{t = \sqrt{\frac{2mz_0^3}{q^2}} \int_\infty^1 \frac{dx}{x^2\sqrt{x-1}}}$$

$u = \frac{z}{z_0}$ $\quad du = \frac{dz}{z_0}$

$$\sqrt{\frac{2m}{q^2}} \int_1^0 \frac{\sqrt{z}\, dz}{\sqrt{1 - \frac{z}{z_0}}}$$

$$t = \sqrt{\frac{2mz_0^3}{q^2}} \int_1^0 \frac{\sqrt{u}\, du}{\sqrt{1-u}}$$

b) since we neglect relativistic effects, the power radiated is just found from the Larmor formula

$$P = \frac{2q^2a^2}{3c^3} \quad \text{where } a = \left|\frac{\vec{F}}{m}\right|$$

well as before, $\vec{F} = -\frac{q^2}{4z^2}\hat{z}$

$$\text{so } a = \frac{q^2}{4mz^2}$$

so the power radiated by the single real point charge is

$$\boxed{P_{real} = \frac{q^4}{6mz^2c^3}}$$

the bound charges in the conductor will also radiate; difficult to say how much, but radiation due to image charge might be a good guess (esp. since total bound charge $= -q$)

$$P_{image} = \frac{q^4}{6mz^2c^3} \qquad \cancel{P_{total}}$$

the total P would be approximately the radiation due to a dipole

dipole moment $p = 2qz$

$$\ddot{p} = 2q\ddot{z} = 2qa$$

$$P = \frac{2\ddot{p}^2}{3c^3} = \frac{8q^2a^2}{3c^3}$$

$$\boxed{\text{so total is } 4 \times P_{real}}$$

c) for the upper region, take the real charge plus a charge of q' at $z=-a$

potential is $V_{up} = \dfrac{q}{\sqrt{r^2+(z-z_0)^2}} + \dfrac{q'}{\sqrt{r^2+(z+a)^2}}$

for the lower region, again take the real charge, but now an image charge of q'' at $z=+b$

potential is $V_{low} = \dfrac{q}{\sqrt{r^2+(z-z_0)^2}} + \dfrac{q''}{\sqrt{r^2+(z-b)^2}}$

now require these potentials to be equal at $z=0$ and require their normal derivatives to obey $\dfrac{\partial V_{up}}{\partial z} = \varepsilon \dfrac{\partial V_{low}}{\partial z}$

$$V_{up}(0) = V_{low}(0) = \frac{q}{\sqrt{r^2+z_0^2}} + \frac{q'}{\sqrt{r^2+a^2}} = \frac{q}{\sqrt{r^2+z_0^2}} + \frac{q''}{\sqrt{r^2+b^2}}$$

so let $q'' = q'$ $\quad b = a$

$$\frac{\partial V_{up}}{\partial z}(0) = +\frac{q}{\cancel{2}(r^2+z_0^2)^{3/2}}(+\cancel{2}z_0) - \frac{q'}{2(r^2+a^2)^{3/2}}(2a)$$

$$\frac{\partial V_{low}}{\partial z}(0) = +\frac{q}{\cancel{2}(r^2+z_0^2)^{3/2}}(-\cancel{2}z_0) - \frac{q'}{2(r^2+a^2)^{3/2}}(-2a)$$

so $$\frac{qz_0}{\cancel{(r^2+z_0^2)^{3/2}}} - \frac{q'a}{\cancel{(r^2+a^2)^{3/2}}} = \varepsilon\left(\frac{qz_0}{\cancel{(r^2+z_0^2)^{3/2}}} + \frac{q'a}{\cancel{(r^2+a^2)^{3/2}}}\right)$$

let $a = z_0$ $\quad q - q' = \varepsilon(q+q')$ $\quad \boxed{F = \dfrac{qq'}{(z_0+a)^2} = \dfrac{(1-\varepsilon)}{(1+\varepsilon)}\dfrac{q^2}{4z_0^2}}$

$$(1-\varepsilon)q = (1+\varepsilon)q'$$

$$q' = \frac{(1-\varepsilon)}{(1+\varepsilon)}q$$