May 2015 Problem E2

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Since there is no legible solution online already that uses separation of variables, I thought I might post one. It is a relatively efficient, clean way to do things.

• Part A

Let $\phi = X(x)\,Y(y)\,Z(z)$. We know immediately that we want $\phi(|z| \to \infty) = 0$. We also know that we want no y dependence. Thus we are limited to the form

$$\phi(x,z) = \mathrm{Re}\left[e^{ikx-k|z|+\gamma}\right] \quad \gamma \in \mathbb{C}\,, \tag{1}$$

while our intuition suggests that we set $\mathrm{Im}[\gamma] = 0$, since this would put the potential out of phase with the charge density, which seems unwanted. This is really up to a factor of π, however, but we will absorb this as $e^{\gamma} = \Gamma$, with $\Gamma \in \mathbb{R}$. Now we recall that given a surface and two approaches $n_{\perp}^{+}$ and $n_{\perp}^{-}$, we know that $E_{\perp}\big|_{n^+}^{n^-} = \frac{\sigma}{\epsilon_0}$ (best to use intuition and not geometry for signs). In our case, we care about the $\hat{\mathbf{z}}$ component when $z = 0$ for arbitrary y, and we will see the matching of functional forms on either side of the equation appear naturally. Namely, we see that.

$$E_z\big|_{z=0^+}^{z=0^-} = 2k\Gamma\cos(kx) = \frac{\sigma_f(x)}{\epsilon} = \frac{\sigma_0}{\epsilon_0}\cos(kx)$$

so we get $\Gamma = \frac{\sigma_0}{2\epsilon_0 k}$. As a sanity check, we see that the $\hat{z}$ component of the field at finite z with $x = 0$ is in the $\hat{\mathbf{z}}$ direction. Plugging in for the field we now get

$$\mathbf{E} = -\nabla\frac{\sigma_0 e^{-k|z|}\cos(kx)}{2\epsilon_0} = \hat{\mathbf{x}}\frac{\sigma_0 e^{-k|z|}\sin(kx)}{2\epsilon_0} + \hat{\mathbf{z}}\frac{\sigma_0 e^{-k|z|}\cos(kx)}{2\epsilon_0}\mathrm{sgn}(z) \tag{2}$$

which seems altogether reasonable.

• Part B

Now the force is rather simple. Generally $\mathbf{F}_{\text{dip.}} = (\mathbf{p}\cdot\nabla)\,\mathbf{E}$ (the directional derivative of the field in the direction of the dipole times the magnitude of the dipole). Now we know that $\mathbf{p} \propto \mathbf{E}$ and that $|\mathbf{p}| = \frac{\alpha\sigma_0 e^{-k|z|}}{2\epsilon_0}$ and the directional derivative along both $\mathbf{p}$ and $\mathbf{E}$ is $\sin(kx)\,\partial_x + \cos(kx)\,\partial_z$ so we find that

$$\mathbf{F} = \left(\frac{\alpha\sigma_0 e^{-k|z|}}{2\epsilon_0}\right)(\sin(kx)\,\partial_x + \cos(kx)\,\partial_z)\left(\hat{\mathbf{x}}\frac{\sigma_0 e^{-k|z|}\sin(kx)}{2\epsilon_0} + \hat{\mathbf{z}}\frac{\sigma_0 e^{-k|z|}\cos(kx)}{2\epsilon_0}\right) = -\hat{z}\frac{k\alpha\sigma_0 e^{-2k|z|}}{4\epsilon_0} \tag{3}$$

which is roughly what one can expect from intuition. I have assumed $z > 0$ here, but it is a simple matter to see that it should not matter and the same result holds for $z < 0$.

• Summary

Problem includes separation of variables, field (discontinuity), and a simple dipole problem.