Problem J99Q.3

(a) By conservation of energy, the electron has kinetic energy

$$E_{\text{kin}} = \hbar\omega - V_0$$

after exiting the well (ignoring the ground-state energy), which corresponds to a momentum

$$p = \sqrt{2mE_{\text{kin}}} = \sqrt{\hbar\omega - V_0}.$$

Note that the continuum wavefunction within the well has a spatial-dependence $e^{ip'x}$ with $p' = \sqrt{2m\hbar\omega}$, and transitions to a dependence e^{ipx} outside the well.

(b) The corotating portion of the drive field gives an interaction Hamiltonian

$$H_{\text{int}} = ex\frac{\epsilon_0}{2}$$

in the rotating-wave approximation, where e is the magnitude of the charge of the electron. The transition matrix element is

$$\begin{aligned}
\langle i|H_{\text{int}}|f\rangle &= \int_0^a \sqrt{\frac{2}{a}}\sin(\pi x/a)\,\frac{\epsilon_0 ex}{2}\,e^{ip'x/\hbar}\,\mathrm{d}x \\
&= \sqrt{\frac{2}{a}}\frac{\epsilon_0 ea^2}{2\pi^2}\int_0^\pi u\sin(u)\,e^{i\alpha u}\,\mathrm{d}u \\
&= \sqrt{\frac{2}{a}}\frac{\epsilon_0 ea^2}{2\pi^2}\frac{e^{i\pi\alpha}(-\pi\alpha^2 - 2i\alpha + \pi) - 2i\alpha}{(\alpha^2-1)^2}
\end{aligned}$$

where $u := \pi x/a$ and $\alpha := p'a/(\pi\hbar)$. In the limit of a very deep well, we have $\alpha \gg 1$ and so

$$\begin{aligned}
\langle i|H_{\text{int}}|f\rangle &\sim -\sqrt{\frac{2}{a}}\frac{\epsilon_0 ea^2}{2\pi}\frac{e^{i\pi\alpha}}{\alpha^2} \\
&= -\sqrt{\frac{2}{a}}\frac{\epsilon_0 e\pi\hbar}{4}\frac{e^{i\pi\alpha}}{m\omega}.
\end{aligned}$$

The density of states at the final continuum state is

$$\rho(E_f) = 2\cdot\frac{1}{2\pi}\frac{\mathrm{d}k}{\mathrm{d}E} = \frac{1}{\hbar\pi}\sqrt{\frac{m}{2E_f}},$$

so Fermi's Golden rule gives

$$\begin{aligned}
\Gamma_{i\to f} &= \frac{2\pi}{\hbar}\left|\langle i|H_{\text{int}}|f\rangle\right|^2\rho(E_f) \\
&= \boxed{\sqrt{\frac{m}{2E_f}}\frac{\epsilon_0^2 e^2\pi^2}{4am^2\omega^2}}.
\end{aligned}$$