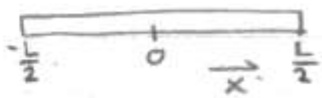
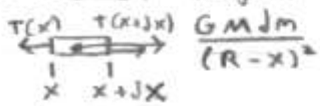
Seth Dorfman Prelims December 30, 2005

January 1999 CM

3) Consider a small piece of rod of length dx





$$(dm)a_r = T(x) - T(x+dx) - \frac{GM(dm)}{(R-x)^2} \qquad dm = \frac{m}{L}dx$$

$$a_r = -\omega^2 r = -\omega^2(R-x)$$

$$\omega^2(R-x)\frac{m}{L} = \frac{T(x+dx)-T(x)}{dx} + \frac{GMm}{(R-x)^2}\cdot\frac{1}{L}$$

$$\frac{dT}{dx} = \omega^2(R-x)\frac{m}{L} - \frac{GMm}{(R-x)^2}\cdot\frac{1}{L}$$

$$T(x) = -\frac{1}{2}\frac{m}{L}\omega^2(R-x)^2 - \frac{GMm}{R-x}\cdot\frac{1}{L} + C_1$$

$$T(\pm\tfrac{L}{2}) = 0 = -\frac{1}{2}\frac{m}{L}\omega^2(R\mp\tfrac{L}{2})^2 - \frac{GMm}{(R\mp L/2)}\cdot\frac{1}{L} + C_1$$

Assume $L \ll R$:

$$0 = -\frac{1}{2}\frac{m}{L}\omega^2R^2\left(1\mp\frac{L}{2R}\right)^2 - \frac{GMm}{R(1\mp L/2R)}\cdot\frac{1}{L} + C_1$$

$$0 = -\frac{1}{2}\frac{m}{L}\omega^2R^2 \pm \frac{m}{L}\omega^2R^2\cdot\frac{L}{2R} - \frac{1}{2}\frac{m}{L}\omega^2R^2\left(\frac{L}{2R}\right)^2$$
$$- \frac{GMm}{RL} \mp \frac{GMm}{RL}\cdot\frac{L}{2R} - \frac{GMm}{RL}\left(\frac{L}{2R}\right)^2 + C_1$$

$$0 = -\frac{1}{2}\frac{m}{L}\omega^2R^2 - \frac{GMm}{RL} \pm \underbrace{\left(\frac{1}{2}m\omega^2R - \frac{1}{2}\frac{GMm}{R^2}\right)}_{=0 \text{ for circular orbit}}$$
$$- \frac{1}{8}mL\omega^2 - \frac{L}{4}\frac{GMm}{R^3} + C_1$$

$$\Rightarrow C_1 = \frac{1}{2}m\omega^2\left(\frac{R^2}{L}+\frac{L}{4}\right) + \frac{GMm}{R}\left(\frac{1}{L}+\frac{L}{4R^2}\right)$$

$$C_1 = \left(\frac{1}{2}m\omega^2 + \frac{GMm}{R^3}\right)\left(\frac{R^2}{L}+\frac{L}{4}\right)$$

$$T(0) = -\frac{1}{2}\frac{m}{L}\omega^2R^2 - \frac{GMm}{R}\cdot\frac{1}{L} + C_1$$
$$= -\left(\frac{1}{2}m\omega^2 + \frac{GMm}{R^3}\right)\frac{R^2}{L} + \left(\frac{1}{2}m\omega^2 + \frac{GMm}{R^3}\right)\left(\frac{R^2}{L}+\frac{L}{4}\right)$$

$$T(0) = \frac{L}{4}\left(\frac{1}{2}m\omega^2 + \frac{GMm}{R^3}\right)$$