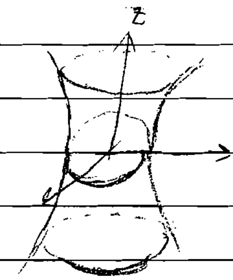
Jan 1999 #2 (CM)

Mass m, surface $x^2+y^2-z^2=R^2$ $\qquad \vec{g}=-g\hat{z}$

cylindrical coordinates: $z^2=\rho^2-R^2$



$T=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\theta}^2+\dot{z}^2)$

$U=mgz$

$g(\rho,z)=\rho^2-z^2-R^2=0$ constraint eqn.

Eliminate z for $\rho \rightarrow$ $z=\pm\sqrt{\rho^2-R^2}$ + for $z>0$, − for $z<0$

$\dot{z}=\frac{\pm\rho\dot{\rho}}{\sqrt{\rho^2-R^2}}$ $\qquad \dot{z}^2=\frac{\rho^2\dot{\rho}^2}{\rho^2-R^2}$ $\qquad \dot{\rho}^2+\dot{z}^2=\dot{\rho}^2\left(1+\frac{\rho^2}{\rho^2-R^2}\right)=\dot{\rho}^2\left(\frac{2\rho^2-R^2}{\rho^2-R^2}\right)$

$$L=T-U=\frac{1}{2}m\left(\rho^2\dot{\theta}^2+\dot{\rho}^2\left[\frac{2\rho^2-R^2}{\rho^2-R^2}\right]\right)\mp mg\sqrt{\rho^2-R^2}$$

$\frac{\partial L}{\partial\theta}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)\Rightarrow$ $m\rho^2\dot{\theta}=\text{constant}\equiv l$ $\qquad \dot{\theta}^2=\frac{l^2}{m^2\rho^4}$

$$\frac{\partial L}{\partial\rho}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\rho}}\right)\Rightarrow m\rho\dot{\theta}^2+\frac{m\dot{\rho}^2}{2}\left[\frac{4\rho}{\rho^2-R^2}-\frac{(2\rho^2-R^2)(2\rho)}{(\rho^2-R^2)^2}\right]\mp\frac{mg\rho}{\sqrt{\rho^2-R^2}}$$

$$=m\ddot{\rho}\left[\frac{2\rho^2-R^2}{\rho^2-R^2}\right]+m\dot{\rho}^2\left[\frac{4\rho}{\rho^2-R^2}-\frac{(2\rho^2-R^2)(2\rho)}{(\rho^2-R^2)^2}\right]$$

$$\ddot{\rho}\left[\frac{2\rho^2-R^2}{\rho^2-R^2}\right]=\frac{l^2}{m^2\rho^3}+\frac{\dot{\rho}^2\rho R^2}{(\rho^2-R^2)^2}\mp\frac{g\rho}{\sqrt{\rho^2-R^2}}$$

• Assume a circular orbit at $\rho=\rho_0$, $z=z_0$ $\quad \dot{\rho}=0$, $\ddot{\rho}=0$

$\Rightarrow \frac{l^2}{m^2\rho_0^3}=\pm\frac{g\rho_0}{\sqrt{\rho_0^2-R^2}}$ $\quad$ [in terms of z_0, $\frac{z_0 l^2}{m^2(z_0^2+R^2)^2}=g$] $\quad$ since ρ_0 is positive, only $z>0$ gives a possible orbit

• Now, assume a small perturbation, $\rho=\rho_0+\eta$, where η is small

$\dot{\rho}=\dot{\eta}$, $\quad \ddot{\rho}=\ddot{\eta}$

$\dot{\rho}^2$ term is second order; negligible

$$\frac{1}{\rho^3}=\frac{1}{(\rho_0+\eta)^3}=\frac{1}{\rho_0^3(1+\frac{\eta}{\rho_0})^3}=\frac{1}{\rho_0^3}\left(1-\frac{3\eta}{\rho_0}\right)=\frac{1}{\rho_0^3}-\frac{3\eta}{\rho_0^4}$$

$$\frac{\rho}{\sqrt{\rho^2-R^2}}\cong(\rho_0+\eta)(\rho_0^2-R^2+2\rho_0\eta)^{-1/2}\approx\frac{(\rho_0+\eta)}{\sqrt{\rho_0^2-R^2}}\left(1+\frac{2\rho_0\eta}{\rho_0^2-R^2}\right)^{-1/2}\approx\frac{(\rho_0+\eta)\left(1-\frac{\rho_0\eta}{\rho_0^2-R^2}\right)}{\sqrt{\rho_0^2-R^2}}$$

$$\frac{\rho}{\sqrt{\rho^2-R^2}} \approx \frac{\rho_0 - \frac{\rho_0^2}{\rho_0^2-R^2}\eta + \eta}{\sqrt{\rho_0^2-R^2}} \approx \frac{\rho_0}{\sqrt{\rho_0^2-R^2}} - \frac{R^2\eta}{(\rho_0^2-R^2)^{3/2}}$$

$$\ddot{\eta}\left[\frac{2\rho^2-R^2}{\rho^2-R^2}\right] = \cancel{\frac{\ell^2}{m^2\rho_0^3}} - \frac{3\ell^2}{m^2\rho_0^4}\eta \mp \cancel{\frac{g\rho_0}{\sqrt{\rho_0^2-R^2}}} \pm \frac{gR^2}{(\rho_0^2-R^2)^{3/2}}\eta$$

$$\ddot{\eta} = -\left(\frac{3\ell^2}{m^2\rho_0^4} \mp \frac{gR^2}{(\rho_0^2-R^2)^{3/2}}\right)\left(\frac{\rho_0^2-R^2}{2\rho_0^2-R^2}\right)\eta$$

frequency of small oscillations:

$$\omega^2 = \left(\frac{3\ell^2}{m^2\rho_0^4} \mp \frac{gR^2}{(\rho_0^2-R^2)^{3/2}}\right)\left(\frac{\rho_0^2-R^2}{2\rho_0^2-R^2}\right)$$

$z = (\rho_0^2-R^2)^{1/2}$ $\qquad \rho_0^2 = z^2+R^2$

$$\Rightarrow \omega^2 = \left[\frac{3\ell^2}{m^2(z_0^2+R^2)^2} - \frac{gR^2}{z_0^3}\right]\left[\frac{z_0^2}{2z_0^2+R^2}\right]$$

Unstable if $\dfrac{gR^2}{z_0^3} > \dfrac{3\ell^2}{m^2(z_0^2+R^2)^2}$ (for $z_0>0$; ~~stable if $z_0<0$~~)

$$\frac{gR^2}{z_0^2} > 3 \underbrace{\frac{z_0\ell^2}{m^2(z_0^2+R^2)^2}}_{3g}$$

$$R^2 > 3z_0^2$$

stable for $0 < z_0 < \dfrac{R}{\sqrt{3}}$