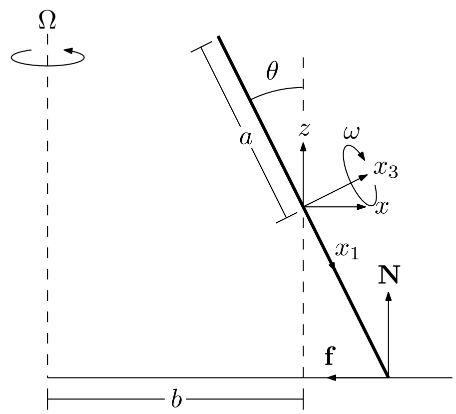
J99M.1 - Rolling Disk (Solution by Jim Wu)

A thin uniform disk of radius a and weight Mg rolls without slipping with constant speed in a circle of radius b on a horizontal plane. The plane of the disk is inclined at an angle θ from the vertical. Find the period τ for the motion around the circle as a function of θ.

Solution:

Let's say that the disk is traveling in a circle in the counterclockwise direction with an angular velocity of $\Omega\hat{\mathbf{z}}$. The disk is also spinning with frequency ω around it's axis of symmetry in the $-\hat{\mathbf{x}}_3$ direction.



So, the angular velocity of the disk at the moment shown in the picture above is

$$\begin{aligned} \boldsymbol{\omega} &= \Omega\hat{\mathbf{z}} - \omega\hat{\mathbf{x}}_3 \\ &= \Omega(\sin\theta\hat{\mathbf{x}}_3 - \cos\theta\hat{\mathbf{x}}_1) - \omega\hat{\mathbf{x}}_3 \\ &= \begin{pmatrix} -\Omega\cos\theta \\ 0 \\ \Omega\sin\theta - \omega \end{pmatrix} \end{aligned} \tag{1}$$

in the basis of the principal moment of inertia axes (body frame). Hence, we can write the angular momentum of the disk as

$$\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega} = \begin{pmatrix} -I_1\Omega\cos\theta \\ 0 \\ I_3(\Omega\sin\theta - \omega) \end{pmatrix} \tag{2}$$

where $\tilde{\mathbf{I}}$ is the moment of inertia tensor, $I_1 = \frac{1}{4}Ma^2$, and $I_3 = \frac{1}{2}Ma^2$. The $\mathbf{L}$ vector precesses around the $\hat{\mathbf{z}}$ axis due to a torque, and this torque will only affect the component of the $\mathbf{L}$ vector in the horizontal plane, which is $\hat{\mathbf{x}}$ in the above picture. Let's compute everything in the fixed frame where $\hat{\mathbf{x}}_1 = \sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}_3 = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\tau} = \left(\frac{d\mathbf{L}}{dt}\right)_{space} &= \boldsymbol{\Omega} \times \mathbf{L} \\
&= \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \times \begin{pmatrix} I_3(\Omega\sin\theta - \omega)\cos\theta - I_1\Omega\cos\theta\sin\theta \\ 0 \\ I_3(\Omega\sin\theta - \omega)\sin\theta + I_1\Omega\cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (I_3 - I_1)\Omega^2\cos\theta\sin\theta - I_3\Omega\omega\cos\theta \\ 0 \end{pmatrix}
\end{aligned} \tag{3}$$

This must be equal to the torque around the center of mass due to the normal force and frictional force. The normal force is $\mathbf{N} = Mg\hat{\mathbf{z}}$ and the frictional force provides the centripetal acceleration of the disk, $\mathbf{f} = -M\Omega^2 b\hat{\mathbf{x}}$. So, the torque about the center of mass is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{pmatrix} a\sin\theta \\ 0 \\ -a\cos\theta \end{pmatrix} \times \begin{pmatrix} -m\Omega^2 b \\ 0 \\ Mg \end{pmatrix} = \begin{pmatrix} 0 \\ M\Omega^2 ba\cos\theta - Mga\sin\theta \\ 0 \end{pmatrix} \tag{4}$$

Equating the torque equations (3) and (4), and applying the no-slip condition $\omega a = \Omega(b + a\sin\theta)$, we can solve for the frequency of precession:

$$\begin{aligned}
M\Omega^2 ba\cos\theta - Mga\sin\theta &= (I_3 - I_1)\Omega^2\cos\theta\sin\theta - I_3\Omega\omega\cos\theta \\
Mga\sin\theta &= \left[-(I_3 - I_1)\sin\theta + I_3\left(\frac{b}{a} + \sin\theta\right) + Mab\right]\Omega^2\cos\theta \\
\Omega^2 &= \frac{Mga\tan\theta}{Mab + I_3\left(\frac{b}{a} + \sin\theta\right) - (I_3 - I_1)\sin\theta} \\
&= \frac{Mga\tan\theta}{Mab + \frac{1}{2}Ma^2\left(\frac{b}{a} + \sin\theta\right) - \frac{1}{4}Ma^2\sin\theta} \\
&= \frac{4g\tan\theta}{6b + a\sin\theta}
\end{aligned} \tag{5}$$

Hence, the period of precession is

$$\boxed{\tau = \frac{2\pi}{\Omega} = \pi\sqrt{\frac{6b + a\sin\theta}{g\tan\theta}}}. \tag{6}$$

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