Jan 1999 #1 (CM)

- Angular velocity in the orbit: \( \omega = \frac{v}{b} \)
- Rolling without slipping: \( \omega = \frac{v}{r} \)
- Moment of inertia of uniform disc about axis of symmetry: \( I = \frac{1}{2}Mr^2 \)
- Penny is at an angle \( \theta \) to the vertical

**Force diagram:**

The friction force, which is what keeps the penny performing circular orbits, is important to not miss or leave out of the problem.

\[ N = mg, \text{ clearly} \]

Circular orbital motion, therefore \( F_c = ma_c = -\frac{mv^2}{b} \)

\[ \Rightarrow F_c = -\frac{mv^2}{b} \]

The angular momentum of the penny about its center of mass separates into a radial and \( \theta \) component:

\[ L = -L\cos \theta - L\sin \theta \]

If the angle \( \theta \) is constant, then the \( \theta \) component of angular momentum is constant, while the radial component precesses.

**Torques about the center of mass:**

Both the normal and friction force act at the contact point with the ground.

Vector \( \vec{r} \) from penny center to ground: \( \vec{r} = \vec{r}_o \sin \theta \hat{r} - \vec{r}_o \cos \theta \hat{z} \)

\[ \vec{N} = \vec{r} \times \vec{F} = -\vec{r}_o \left( \sin \theta \hat{r} - \cos \theta \hat{z} \right) \times \left( -\frac{mv^2}{b} \hat{r} + mg \hat{z} \right) \]

\[ = -mg \vec{r}_o \sin \theta \hat{\theta} + \frac{mv^2 \vec{r}_o \cos \theta \hat{\phi}}{b} \]

\[ \vec{N} = \frac{dL}{dt} = -L\sin \theta \hat{\theta} \]

\[ \Rightarrow N_2 = 0 \quad (\text{This proves } L = \text{ constant}) \]

\[ \frac{d}{dt} \left( -L\cos \theta \hat{\phi} \right) = -L \cos \phi \frac{d}{dt} (\hat{\phi}) = -L \cos \theta \hat{\phi} \]

Orbital angular velocity, \( \hat{\phi} \), which \( \hat{\phi} \) precesses
\[ -L \frac{V}{b} \cos \theta = -mg \rho \sin \theta + \frac{mv^2 r_0}{b} \cos \theta \]

\[ mg \rho \tan \theta = \frac{Lv}{b} + \frac{mv^2 r_0}{b} \]

\[ L = I \omega = \frac{1}{2} m r_0^2 \omega = \frac{1}{2} m r_0 V \]

\[ mg \rho \tan \theta = \frac{1}{2} \frac{m r_0 v^2}{b} + \frac{mv^2 r_0}{b} \]

\[ \tan \theta = \frac{3v^2}{2bg} \]

\[ v^2 = \frac{2b \tan \theta}{3} \]

\[ \tau = \frac{2 \pi b}{v} = \frac{2 \pi b \sqrt{3}}{\sqrt{2bg \tan \theta}} = \frac{\pi \sqrt{6b}}{\sqrt{2bg \tan \theta}} \]