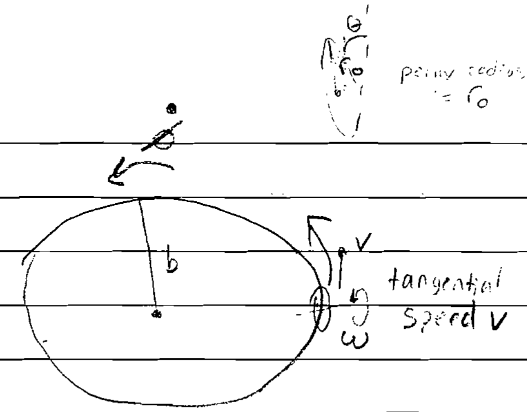
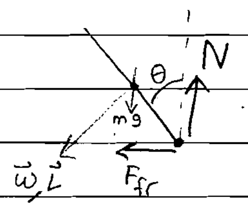
Jan 1999 #1 (CM)



- Angular velocity in the orbit: $\dot{\phi} = \frac{v}{b}$
- Rolling without slipping: $\omega = \frac{v}{r_o}$
- Moment of inertia of uniform disc about axis of symmetry: $I = \frac{1}{2} m r_o^2$
- penny is at an angle θ to the vertical

Force diagram:



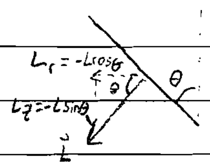
The friction force, which is what keeps the penny performing circular orbits, is important to not miss or leave out of the problem.

$N = mg$, clearly

Circular orbital motion, therefore $F_r = m a_r \cong -\frac{mv^2}{b}$

$\Rightarrow \vec{F}_{fr} = \frac{-mv^2}{b}\hat{r}$

The angular momentum of the penny about its center of mass separates into a radial and z component:



$\vec{L} = -L\cos\theta\,\hat{r} - L\sin\theta\,\hat{z}$

If the angle θ is constant, then the z component of angular momentum is constant, while the radial component precesses.

Torques about the center of mass?

Both the Normal and friction force act at the contact point with the ground

vector $\vec{r}$ from penny CoM to ground: $\vec{r} = r_o\sin\theta\,\hat{r} - r_o\cos\theta\,\hat{z}$

$\vec{N} = \vec{r}\times\vec{F} = r_o(\sin\theta\,\hat{r} - \cos\theta\,\hat{z}) \times \left(-\frac{mv^2}{b}\hat{r} + mg\,\hat{z}\right)$

$= -mgr_o\sin\theta\,\hat{\phi} + \frac{mv^2 r_o\cos\theta}{b}\hat{\phi}$

$\vec{N} = \frac{d\vec{L}}{dt}$ $\quad \frac{dL_z}{dt} = -\dot{L}\sin\theta\,\hat{z}$, but $N_z = 0$ (This proves $\dot{L} = 0$; $|\vec{L}| = \text{constant}$)

$\frac{d}{dt}(-L\cos\theta\,\hat{r}) = -L\cos\theta\frac{d}{dt}(\hat{r}) = -L\cos\theta\,\dot{\phi}\,\hat{\phi}$

orbital angular velocity, at which $\hat{r}$ precesses

$\dot{\phi} = \frac{v}{b}$

$$\Rightarrow -L\frac{v}{b}\cos\theta = -mgr_0\sin\theta + \frac{mv^2 r_0}{b}\cos\theta$$

$$mgr_0\tan\theta = \frac{Lv}{b} + \frac{mv^2 r_0}{b} \qquad L = I\omega = \frac{1}{2}mr_0^2\omega = \frac{1}{2}mr_0 v$$

$$mgr_0\tan\theta = \frac{1}{2}\frac{mr_0v^2}{b} + \frac{mv^2 r_0}{b}$$

$$\Rightarrow \tan\theta = \frac{3v^2}{2bg}$$

$$v^2 = \frac{2bg\tan\theta}{3}$$

$$\tau = \frac{2\pi b}{v} = \frac{2\pi b\sqrt{3}}{\sqrt{2bg\tan\theta}} = \pi\sqrt{\frac{6b}{g\tan\theta}}$$