Problem J99E.2

There are four fields at play:
1. The $E$-field of the plane wave,
2. The $B$-field of the plane wave,
3. The $E$-field of the charge,
4. The $B$-field of the charge.

The momentum we seek must be proportional to a cross-term between an $E$ and a $B$ field, since
$$\mathbf{P} = \frac{S}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0 c^2}.$$  

Moreover, since the electron’s motion (and thus its mechanical momentum) is linear in $E_0$, the cross-term responsible for the transverse momentum must also be linear in $E_0$.

Taylor expanding the aforementioned four fields with respect to $E_0$, we obtain:

<table>
<thead>
<tr>
<th>Field</th>
<th>Static Component</th>
<th>Linear Component $(\propto E_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{wave}}$</td>
<td>0</td>
<td>$E_0 \cos(kz - \omega t) \hat{x}$</td>
</tr>
<tr>
<td>$B_{\text{wave}}$</td>
<td>0</td>
<td>$c^{-1}E_0 \cos(kz - \omega t) \hat{y}$</td>
</tr>
<tr>
<td>$E_{\text{charge}}$</td>
<td>$e/(4\pi\epsilon_0 r^2) \hat{r}$</td>
<td>something</td>
</tr>
<tr>
<td>$B_{\text{charge}}$</td>
<td>0</td>
<td>something</td>
</tr>
</tbody>
</table>

The only two cross-terms with linear scaling in $E_0$ are $E_{\text{charge}}\times B_{\text{charge}}$ and $E_{\text{charge}}\times B_{\text{wave}}$. The former term is an electron self-interaction energy, and is ill-behaved (renormalized away).

Thus we compute
$$\int \mathbf{P} \, dV = \frac{1}{\mu_0 c^2} \int (E_{\text{charge}}\times B_{\text{wave}}) \, dV$$
$$= \frac{eE_0}{4\pi c} \int r^{-2} \cos(kz - \omega t) \, (\hat{r} \times \hat{y}) \, dV$$
$$= \frac{eE_0}{4\pi c} \, \text{Re} \left\{ e^{-i\omega t} \int r^{-2} e^{ikz} \, (\hat{r} \times \hat{y}) \, dV \right\}.$$  

The integral expands to
$$\int r^{-2} e^{ikz} \, (\hat{r} \times \hat{y}) \, dV = \int r^{-3} e^{ikz} \, (x \hat{z} - z \hat{x}) \, dV$$
$$= -i\hat{x} \int r^{-3} z \sin(kz) \, dV,$$

where we have eliminated the terms odd in $x, z$, since they integrate to zero. In spherical coordinates, we evaluate
$$I := \int r^{-3} z \sin(kz) \, dV$$
$$= 2\pi \int_0^\infty \frac{1}{r^3} \int_0^\pi (r \cos \theta) \sin(kr \cos \theta) r^2 \sin \theta \, d\theta \, dr$$
$$= 2\pi \int_0^\infty \int_{-1}^1 u \sin(kru) \, du \, dr.$$
where \( u := \cos \theta \). Integration by parts yields

\[
\int_{-1}^{1} u \sin(kru) \, du = - \left[ \frac{u \cos(kru)}{k r} \right]_{u=-1}^{1} + \frac{1}{kr} \int_{-1}^{1} \cos(kru) \, du
\]

\[
= - \frac{2 \cos(kr)}{kr} + \frac{1}{(kr)^2} [\sin(kru)]_{u=-1}^{1},
\]

\[
= - \frac{2 \cos(kr)}{kr} + \frac{2 \sin(kr)}{(kr)^2}
\]

\[
= - \frac{2}{k^2} \partial_r (r^{-1} \sin(kr)),
\]

so that

\[
I = - \frac{4\pi}{k^2} \int_0^{\infty} \partial_r (r^{-1} \sin(kr)) \, dr = \frac{4\pi}{k}.
\]

Combining the above, we have

\[
\int P \, dV = - \frac{eE_0}{4\pi c} \sin(\omega t) \, I = - \frac{eE_0}{\omega} \sin(\omega t).
\]

For comparison, the trajectory of the particle satisfies

\[
\dot{p}_x = eE_0 \cos(kz - \omega t) \approx eE_0 \cos(\omega t)
\]

assuming transverse motion, which yields

\[
p_x = \frac{eE_0}{\omega} \sin(\omega t)
\]

in the frame where the electron has zero average velocity. This is precisely the negative of the field momentum, so Newton’s third law is satisfied.