Problem J99E.2

There are four fields at play:

1. The E-field of the plane wave,
2. The B-field of the plane wave,
3. The E-field of the charge,
4. The B-field of the charge.

The momentum we seek must be proportional to a cross-term between an E and a B field, since

$$\mathbf{P} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0 c^2}.$$

Moreover, since the electron's motion (and thus its mechanical momentum) is linear in E_0, the cross-term responsible for the transverse momentum must also be linear in E_0.

Taylor expanding the aforementioned four fields with respect to E_0, we obtain:

Field	Static Component	Linear Component ($\propto E_0$)
$\mathbf{E}_{\text{wave}}$	0	$E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}$
$\mathbf{B}_{\text{wave}}$	0	$c^{-1} E_0 \cos(kz - \omega t)\,\hat{\mathbf{y}}$
$\mathbf{E}_{\text{charge}}$	$e/(4\pi\epsilon_0 r^2)\,\hat{\mathbf{r}}$	something
$\mathbf{B}_{\text{charge}}$	0	something

The only two cross-terms with linear scaling in E_0 are $\mathbf{E}_{\text{charge, static}} \times \mathbf{B}_{\text{charge}}$ and $\mathbf{E}_{\text{charge, static}} \times \mathbf{B}_{\text{wave}}$. The former term is an electron self-interaction energy, and is ill-behaved (renormalized away).

Thus we compute

$$\begin{aligned}
\int \mathbf{P}\, \mathrm{d}V &= \frac{1}{\mu_0 c^2} \int (\mathbf{E}_{\text{charge, static}} \times \mathbf{B}_{\text{wave}})\, \mathrm{d}V \\
&= \frac{eE_0}{4\pi c} \int r^{-2} \cos(kz - \omega t)\, (\hat{\mathbf{r}} \times \hat{\mathbf{y}})\, \mathrm{d}V \\
&= \frac{eE_0}{4\pi c} \,\mathrm{Re}\left\{ e^{-i\omega t} \int r^{-2} e^{ikz}\, (\hat{\mathbf{r}} \times \hat{\mathbf{y}})\, \mathrm{d}V \right\}.
\end{aligned}$$

The integral expands to

$$\begin{aligned}
\int r^{-2} e^{ikz}\, (\hat{\mathbf{r}} \times \hat{\mathbf{y}})\, \mathrm{d}V &= \int r^{-3} e^{ikz}\, (x\,\hat{\mathbf{z}} - z\,\hat{\mathbf{x}})\, \mathrm{d}V \\
&= -i\hat{\mathbf{x}} \int r^{-3} z \sin(kz)\, \mathrm{d}V\,,
\end{aligned}$$

where we have eliminated the terms odd in x, z, since they integrate to zero. In spherical coordinates, we evaluate

$$\begin{aligned}
I &:= \int r^{-3} z \sin(kz)\, \mathrm{d}V \\
&= 2\pi \int_0^\infty \frac{1}{r^3} \int_0^\pi (r\cos\theta) \sin(kr\cos\theta)\, r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}r \\
&= 2\pi \int_0^\infty \int_{-1}^{1} u \sin(kru)\, \mathrm{d}u\, \mathrm{d}r
\end{aligned}$$

where $u := \cos\theta$. Integration by parts yields

$$\begin{aligned}
\int_{-1}^{1} u \sin(kru)\, \mathrm{d}u &= -\left[\frac{u\cos(kru)}{kr}\right|_{u=-1}^{1} + \frac{1}{kr}\int_{-1}^{1} \cos(kru)\, \mathrm{d}u \\
&= -\frac{2\cos(kr)}{kr} + \frac{1}{(kr)^2}\left[\sin(kru)\right|_{u=-1}^{1}, \\
&= -\frac{2\cos(kr)}{kr} + \frac{2\sin(kr)}{(kr)^2} \\
&= -\frac{2}{k^2}\,\partial_r(r^{-1}\sin(kr)),
\end{aligned}$$

so that

$$I = -\frac{4\pi}{k^2}\int_0^\infty \partial_r(r^{-1}\sin(kr))\ \mathrm{d}r = \frac{4\pi}{k}.$$

Combining the above, we have

$$\int \mathbf{P}\,\mathrm{d}V = -\frac{eE_0}{4\pi c}\sin(\omega t)\, I = -\frac{eE_0}{\omega}\sin(\omega t).$$

For comparison, the trajectory of the particle satisfies

$$\dot{p}_x = eE_0\cos(kz - \omega t) \approx eE_0\cos(\omega t)$$

assuming transverse motion, which yields

$$p_x = \frac{eE_0}{\omega}\sin(\omega t)$$

in the frame where the electron has zero average velocity. This is precisely the negative of the field momentum, so Newton's third law is satisfied.