1 Dec 2021

J98T.2—Cooling Liquid Helium

Problem

Consider a closed dewar containing liquid ^{4}He (whose atoms are spin zero bosons for our purposes) in equilibrium with its vapor at low temperatures.

a) The latent heat of vaporization per atom of ^{4}He is l at $T = 0$ which fixes the chemical potential. What is the vapor pressure at temperatures $k_B T \ll l$? You may neglect the temperature dependence of the chemical potential and make other reasonable approximations.

b) ^{4}HE at one atmosphere of pressure boils at about 4K. Use your result from part a) to get a *rough* estimate of l based on this datum.

c) The dewar is not perfectly insulating whence heat leaks into the liquid ^{4}He at a rate $\dot{Q}$. At what rate $\dot{V}$ (volume per unit time) doe a pump have to remove the vapor to keep the (low) temperature from rising? (Pumping is a simple means of cooling liquid ^{4}He.)

Useful numbers:

$$
\begin{aligned}
m_{\text{He}} &\approx (2/3) \times 10^{-23} \text{ g} \\
k_B &= 1.3807 \times 10^{-23} \text{ J/K} \\
h &= 6.6262 \times 10^{-34} \text{J s}
\end{aligned}
$$

J98T.2

a) $\dfrac{dP}{dT} = \dfrac{\ell}{T\Delta v} \xrightarrow[\frac{V_g}{N} = \frac{P}{KT}]{\Delta v \equiv \frac{V_g}{N}} \dfrac{dP}{dT} = \dfrac{\ell P}{KT^2}$

$$\frac{dP}{P} = \frac{\ell}{K}\frac{dT}{T^2} \rightarrow \ln P = -\frac{\ell}{KT} + C$$

$\boxed{P = P_0 e^{-\ell/KT}}$ ← P_0 = high-temp limit of P

b) $1 = P_0 e^{-\ell/4K}$

$\boxed{\ell = 4K \ln P_0}$ ← $[P_0]$ = atm

c) $\Delta H = TdS = dU + PdV \leftarrow dU \Rightarrow dU + \dot{Q}\,dt$

$$T = \frac{dU}{dS} + P\frac{dV}{dS} + \dot{Q}\frac{dt}{dS} \leftarrow \frac{dt}{dS} = \frac{1}{\dot{V}}\frac{dV}{dS}$$

$$= T + \left(P + \frac{\dot{Q}}{\dot{V}}\right)\frac{dV}{dS}$$

$$\left(P + \frac{\dot{Q}}{\dot{V}}\right)\frac{dV}{dS} = 0$$

$$\frac{\dot{Q}}{\dot{V}} = -P = -P_0 e^{-\ell/KT}$$

$$\boxed{\dot{V} = -\frac{\dot{Q}}{P_0} e^{\ell/KT}}$$

Scratch work for a)

$dH \equiv N\ell = dU + PdV$

- This led down multiple rabbit holes

$G = U - TS + PV = \mu N$

$G = (U_g + U_\ell) - T(S_G + S_\ell) + P(V_g + V_\ell) = \mu N$

$U_g = \frac{3}{2} N_g KT$

$T\Delta S = N\ell$

$\Delta S = S_G - S_\ell$

assume $S_\ell << S_G$

$U_\ell = U_g - N_g \ell$

$= 3N_g KT - N_g \ell - N_g \ell + PV_g + PV_\ell = \mu N$

$\mu N = N_g(3KT - 2\ell) + PV$

- what...?

Scratch work for b)

Scratch work for c)

$\Delta H = TdS$

$T = \frac{dH}{dS} \leftarrow dH = \ell = dU + PdV \leftarrow dU \rightarrow dU + \dot{Q} dt$

$= \frac{dU}{dS} + P\frac{dV}{dS} + \dot{Q}\frac{dt}{dS} = \frac{dU}{dS} + P\frac{dV}{dS} + \frac{\dot{Q}}{\dot{V}}\frac{dV}{dS}$

$T = T + \left(P + \frac{\dot{Q}}{\dot{V}}\right)\frac{dV}{dS}$

$P + \frac{\dot{Q}}{\dot{V}} = 0 \rightarrow \dot{V} = -\dot{Q}/P = -\frac{\dot{Q}}{P_0} e^{(\ell/KT)}$