January 1998 Preliminary Exam, Quantum Mechanics Problem 3

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Problem (Hyperfine Structure):

The hyperfine structure of the $n = 1$ level of hydrogen arises from a coupling between the electron and proton spins of the form

$$H_{hyperfine} = a\vec{s}_e \cdot \vec{s}_p \tag{1}$$

where a is a positive constant. The other terms in the hydrogen atom Hamiltonian do not lift the degeneracy of the $n = 1$ level and may be ignored in this problem

(a) Find the energies and degeneracies of the $n = 1$ hyperfine levels.

A uniform magnetic field $\vec{B}$ is switched on for a period of time. For simplicity, assume that the field is constant for $0 < t < T$ and zero at all other times.

(b) To a very good approximation, we ignore the magnetic interaction of the proton with the field compared to that of the electron. Briefly explain why.

(c) Given that the atom was in its ground state before the magnetic field was turned on, what is the probability that it is in its ground state for $t > T$?

Solution:

(a) Choose quantum numbers $S^2 = (\vec{s}_e + \vec{s}_p)^2$ and S_z. Rewrite the Hamiltonian dot product in this new form:

$$H_{hyperfine} = \frac{a}{2}\left(S^2 - s_e^2 - s_p^2\right) = \frac{a}{2}\left(S^2 - \frac{3}{2}\hbar^2\right) \tag{2}$$

S^2 has eigenvalue $s(s+1)\hbar^2$, where the composite spin value $s \in \{0, 1\}$. The energy corresponding to eigenvalue s is $2s + 1$-fold degenerate. Therefore,

$$\langle s = 0, m = 0| \, H \, |s = 0, m = 0\rangle = \frac{a}{2}\left(0 - \frac{3}{2}\hbar^2\right) = \frac{-3}{4}a\hbar^2 \equiv E_0 \tag{3}$$

$$\langle s = 1, m = 0, \pm 1| \, H \, |s = 1, m = 0, \pm 1\rangle = \frac{a}{2}\left(2\hbar^2 - \frac{3}{2}\hbar^2\right) = \frac{1}{4}a\hbar^2 \equiv E_1 \tag{4}$$

E_0 is non-degenerate, and E_1 has 3-fold degeneracy.

(b) With a magnetic field $\vec{B}$, the Hamiltonian gains two additional factors, as follows:

$$H'_{hyperfine} = a\vec{s}_e \cdot \vec{s}_p - \vec{\mu}_e \cdot \vec{B} - \vec{\mu}_p \cdot \vec{B} \tag{5}$$

$$\vec{\mu} = \frac{gq}{2m}\vec{S} \tag{6}$$

For both the electron and the proton, $g \approx 2$. However, $m_p \approx 1840 m_e$. Therefore, the electron magnetic field term is several orders of magnitude larger than the proton magnetic field term. Thus, it is a very good approximation to simply ignore the proton magnetic field term.

(c) Without loss of generality, we can assume $\vec{B} = B\hat{z}$. Therefore,

$$H'_{hyperfine} \approx a\vec{s}_e \cdot \vec{s}_p - \mu_B B \sigma_e^z \tag{7}$$

where $\mu_B \equiv \frac{e\hbar}{2m_e}$ and $S_z = \frac{\hbar}{2}\sigma_e^z$. The ground state wave function, where $s = 0$ and $m = 0$, can be written explicitly in the z-oriented basis, as follows:

$$|0,0,t=0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_e |\downarrow\rangle_p - |\downarrow\rangle_e |\uparrow\rangle_p] \tag{8}$$

This state will mix with the symmetric state $|1,0,t=0\rangle$, as follows:

$$|1,0,t=0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_e |\downarrow\rangle_p + |\downarrow\rangle_e |\uparrow\rangle_p] \tag{9}$$

The other two states, $|1,1,t=0\rangle$ and $|1,-1,t=0\rangle$, are still stationary states of this new Hamiltonian and, thus, will not mix with the previous two states. We first change the basis of the σ_e^z to the total angular momentum basis using the Clebsch Gordan coefficients:

$$\begin{aligned} -\mu_B B\sigma_e^z = -\mu_B B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &\to -\mu_B B \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= -\mu_B B \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = -\mu_B B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{10}$$

$$\implies H'_{hyperfine,red} = \begin{bmatrix} \frac{1}{4}a\hbar^2 & -\mu_B B \\ -\mu_B B & \frac{-3}{4}a\hbar^2 \end{bmatrix} \equiv \begin{bmatrix} \alpha & -\beta \\ -\beta & -3\alpha \end{bmatrix} = -\alpha I + 2\alpha\sigma^z - \beta\sigma^x \tag{11}$$

where $\alpha \equiv \frac{1}{4}a\hbar^2$ and $\beta \equiv \mu_B B$. To evolve wavefunctions, I have put the Hamiltonian in terms of the Pauli spin matrices to use the following identity:

$$e^{ia(\hat{n}\cdot\vec{\sigma})} = I\cos(a) + i(\hat{n}\cdot\vec{\sigma})\sin(a) \tag{12}$$

The normalization we use is $\sqrt{(2\alpha)^2 + \beta^2} = \sqrt{4\alpha^2 + \beta^2} \equiv \gamma$. Therefore,

$$\begin{aligned} \langle 0,0,t=0|0,0,t\rangle &= \langle 0,0,t=0|\, exp\left(-\frac{i}{\hbar}Ht\right)|0,0,t=0\rangle \\ &= \langle 0,0,t=0|\, exp\left(\frac{i}{\hbar}\gamma t(\frac{\alpha}{\gamma}I - 2\frac{\alpha}{\gamma}\sigma^z + \frac{\beta}{\gamma}\sigma^x)\right)|0,0,t=0\rangle \\ &= e^{\frac{i}{\hbar}\alpha t}\langle 0,0,t=0|\, exp\left(\frac{i}{\hbar}\gamma t(-2\frac{\alpha}{\gamma}\sigma^z + \frac{\beta}{\gamma}\sigma^x)\right)|0,0,t=0\rangle \\ &= e^{\frac{i}{\hbar}\alpha t}\langle 0,0,t=0|\, I\cos\Big(\frac{\gamma}{\hbar}t\Big) + i(\frac{-2\alpha}{\gamma}\sigma^z + \frac{\beta}{\gamma}\sigma^x)\sin\Big(\frac{\gamma}{\hbar}t\Big)|0,0,t=0\rangle \\ &= e^{\frac{i}{\hbar}\alpha t}(\cos\Big(\frac{\gamma}{\hbar}t\Big) + i\frac{2\alpha}{\gamma}\sin\Big(\frac{\gamma}{\hbar}t\Big)) \end{aligned} \tag{13}$$

Here, I used the fact that the $\langle 0,0,t=0|$ and $|0,0,t=0\rangle$ will pick out the bottom right term of the Hamiltonian.

$$\implies |\langle 0,0,t=0|0,0,t\rangle|^2 = \cos^2(\frac{\gamma}{\hbar}t) + \frac{4\alpha^2}{\gamma^2}\sin^2\left(\frac{\gamma}{\hbar}t\right) = 1 - \frac{\beta^2}{\gamma^2}\sin^2\left(\frac{\gamma}{\hbar}t\right) \tag{14}$$

Evaluate at time $t = T$:

$$\implies |\langle 0,0,t=0|0,0,t\rangle|^2 = 1 - \frac{\beta^2}{\gamma^2}\sin^2\left(\frac{\gamma}{\hbar}T\right) = 1 - \frac{4\mu_B^2 B^2}{a^2\hbar^4 + 4\mu_B^2 B^2}\sin^2\left(\frac{T}{\hbar}\sqrt{\frac{1}{4}a^2\hbar^4 + \mu_B^2 B^2}\right) \tag{15}$$