Problem J98Q.3

(a) We may rewrite

$$H_{\text{hyperfine}} = \frac{a}{2}\left(F^2 - s_e^2 - s_p^2\right) = \frac{a}{2}\left(F^2 - 3/2\right),$$

where $F := s_e + s_p$. Using the appropriate Clebsch-Gordan coefficients, we have a singlet state

$$|F=0, m_F=0\rangle = \frac{|m_e = 1/2\rangle \otimes |m_p = -1/2\rangle - |m_e = -1/2\rangle \otimes |m_p = 1/2\rangle}{\sqrt{2}}$$

with energy $-3a/4$, where m_e, m_s denote the projection of the spins s_e, s_p along the B-field quantization axis. We also have a threefold degenerate triplet state

$$\begin{aligned} |F=1, m_F=-1\rangle &= |m_e = -1/2\rangle \otimes |m_p = -1/2\rangle, \\ |F=1, m_F=0\rangle &= \frac{|m_e = 1/2\rangle \otimes |m_p = -1/2\rangle + |m_e = -1/2\rangle \otimes |m_p = 1/2\rangle}{\sqrt{2}}, \\ |F=1, m_F=1\rangle &= |m_e = 1/2\rangle \otimes |m_p = 1/2\rangle, \end{aligned}$$

with energies $a/4$. As expected, the average energy is zero.

(b) Since the mass of the proton is $\sim 1800\times$ heavier than the electron, its magnetic moment is correspondingly smaller than that of the electron. Thus the magnetic interaction Hamiltonian

$$H_{\text{magnetic}} = -g_e \mu_B s_e \cdot B - g_N \mu_N s_p \cdot B$$

is dominated by the μ_B term.

(c) In the basis $|F=0, m_F=0\rangle, |F=1, m_F=1\rangle$, the effective Hamiltonian is

$$H_{\text{eff}} = \begin{bmatrix} a/4 & g\mu_B B/2 \\ g\mu_B B/2 & -3a/4 \end{bmatrix} = -\frac{a}{4}\mathbf{I} + \frac{a}{2}\sigma_z + \frac{g\mu_B B}{2}\sigma_x.$$

During the evolution time T, the field therefore drives Rabi oscillations at the generalized Rabi frequency

$$\tilde{\Omega} = \sqrt{\Delta^2 + \Omega^2}, \quad \text{where} \quad \Delta = a/2 \quad \text{and} \quad \Omega = \frac{g\mu_B B}{2}.$$

The desired excitation probabliity is thus

$$P_{g \to e} = \boxed{\frac{\Omega^2}{\tilde{\Omega}^2}\sin^2(\tilde{\Omega}T).}$$

Time: 17 m 16 s