Problem J98Q.2

(a) The bound state must be of the form

$$\Psi(x) = \begin{cases} Ae^{k(L+x)} & x < -L, \\ Be^{-k(L+x)} + Ce^{-k\,(L-x)} & -L < x < L, \\ De^{-k\,(x-L)} & x > L \end{cases}$$

for some $k > 0$, with corresponding energy

$$E = -\frac{\hbar^2 k^2}{2m}.$$

Imposing continuity at $x = \pm L$ yields

$$A = B + \alpha C \quad \text{and} \quad D = C + \alpha B, \tag{*}$$

where $\alpha := e^{-2kL}$. Moreover, we have the boundary conditions

$$\Psi'(-L^+) - \Psi'(-L^-) = -\frac{2ma}{\hbar^2}\Psi(-L) \quad \text{and} \quad \Psi'(L^+) - \Psi'(L^-) = \frac{2ma}{\hbar^2}\Psi(L)$$

obtained by integrating the Schrödinger equation. These conditions reduce to

$$-B + \alpha C - A = -\beta A \quad \text{and} \quad D + C - \alpha B = -\beta D,$$

where

$$\beta := \frac{2}{k\ell} \quad \text{and} \quad \ell := \frac{\hbar^2}{ma}.$$

Combining this with (*) yields the conditions

$$(1-\beta)(B+\alpha C) + B - \alpha C = 0 \quad \text{and} \quad (1+\beta)(C+\alpha B) + C - \alpha B = 0,$$

which reduce to

$$(2-\beta)B - (1+\beta)\,\alpha C = 0 \quad \text{and} \quad (2+\beta)C - (1-\beta)\,\alpha B = 0.$$

Combining these equations give the transcendental relation

$$(4-\beta^2) = (1-\beta^2)\,\alpha^2,$$

which rearranges to

$$k^2\ell^2 = \frac{4\,(1-\alpha^2)}{4-\alpha^2}. \tag{**}$$

This expands into the desired relation between E, m, a, and L.

As expected, for $L \to \infty$, we have $\alpha \to 0$ and so we recover the result $k = 1/\ell$ for a single delta function.

(b) For small L, the equation (**) becomes

$$k^2\ell^2 = \frac{16kL}{3}$$

to leading-order, which gives

$$k = \frac{16L}{3\ell^2}.$$

It follows that

$$E = -\frac{128\hbar^2 L^2}{9m\ell^4} = -\frac{128}{9}\frac{aL^2}{\ell^3}$$

to leading-order, as desired.