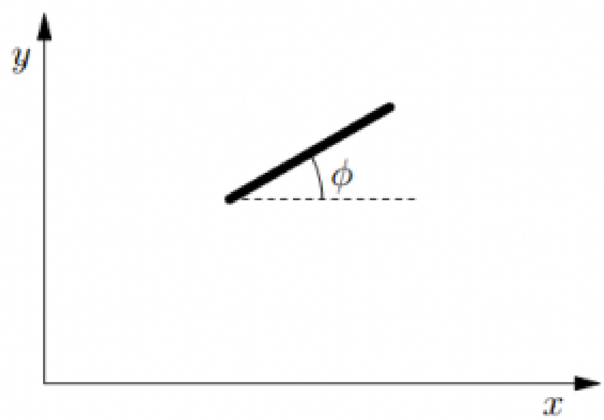
J98M.3

Problem:

As a simplified model for the motion of a skate, assume that the blade experiences no friction when it moves along itself and/or turns around its center. The blade cannot move translational normal to itself. Now consider a skate moving on an icy inclined plane which makes a 30 degree angle with the horizontal. In view of the assumption above, you may think of the blade as a thin uniform rod of mass M moving on the plane under the influence of gravity subject to the constraint that it cannot move translational normal to itself. Introduce Cartesian coordinates x and y on the plane, with x pointing down the incline. The blade is characterized by its center of mass position (x, y), and the angle ϕ it makes with the x-axis.

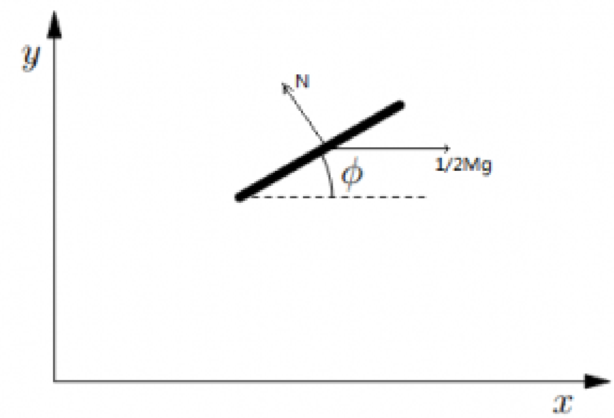


1. Write down the equations of motion including the reaction force normal to the blade.
2. Write down the constraint on the motion in terms of x, y, ϕ, and their time derivatives.
3. At time $t = 0$; $x = y = \phi = \dot{x} = \dot{y} = 0$ and $\dot{\phi} = \omega$. Find the subsequent trajectory. Hint: The reaction force normal to the blade is proportional to $\sin \omega t$.

Solution:

Equations of Motion

The forces exerting on the blade are as shown in Figure 2. Note that there are no net torque on the blade, the ϕ coordinate just undergoes free motion. Given that the initial rotation around the center has angular velocity ω, we can easily write down:



$$\phi = \omega t \tag{1}$$

$$\ddot{\phi} = 0 \tag{2}$$

Note that since we are on an incline of 30 degrees, the gravational force is only $\frac{1}{2}Mg$. Next, simply apply Newton's second law, we write down the EOMs with normal force N:

$$M\ddot{x} = \frac{1}{2}Mg - N\sin\omega t \tag{3}$$

$$M\ddot{y} = N\cos\omega t \tag{4}$$

Constraint of motion

The only constraint on the motion is that the blade cannot move translational normal to itself. Such an constraint can be written as the relation between the x direction and y direction velocity:

$$\frac{\dot{y}}{\dot{x}} = \tan\phi = \tan\omega t \tag{5}$$

With the EOMs written, it remains to solve them. Take $N = A \sin \omega t$ (suggested by Hint), where A is the normal force amplitude to be determined, we have:

$$\ddot{x} = \frac{1}{2} g - \frac{A}{M} \sin^2 \omega t \tag{6}$$

$$\ddot{y} = \frac{A}{M} \sin \omega t \cos \omega t \tag{7}$$

Integrate (6) and (7) once, we have:

$$\dot{x} = \frac{1}{2} gt - \frac{At}{2M} + \frac{A \sin 2\omega t}{4\omega M} \tag{8}$$

$$\dot{y} = \frac{A(1 - \cos 2\omega t)}{4\omega M} \tag{9}$$

Using (5), we solve for A:

$$\frac{1}{2} gt - \frac{At}{2M} + \frac{A \sin 2\omega t}{4\omega M} = \frac{A(1 - \cos 2\omega t)}{4\omega M \tan \omega t} \tag{10}$$

It's easy to see that $A = Mg$. Thus the normal force $N = Mg \sin \omega t$.

Now integrate (8) and (9) to obtain the trajectory.

$$x = \frac{g}{8\omega^2} (1 - \cos 2\omega t) \tag{11}$$

$$y = \frac{g}{4\omega^2} (2\omega t - \sin 2\omega t) \tag{12}$$

One thought on “J98M.3”

October 17, 2013 at 5:46 pm

Good, correct solution.

But you missed an extra factor of $1/2$ in (12). Check it one more time.