J98M.3

To begin, we see that the kinetic energy of the blade is

$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \tag{1}$$

while the potential energy is given by

$$U = -\cos 30^\circ mgx = -\frac{1}{2} mgx \tag{2}$$

However, we must deal with the constraint

$$\dot{y} = \dot{x} \tan \phi. \tag{3}$$

The equations of motion must then be

$$I \ddot{\phi} = 0 \tag{4}$$

$$m \ddot{x} = F_{\text{constraint},x} + \frac{1}{2} mg \tag{5}$$

$$m \ddot{y} = F_{\text{constraint},y} \,. \tag{6}$$

Note here that since $\dot{\phi}$ is a constant, we automatically assume solution of

$$\phi = \omega t + \alpha \tag{7}$$

where $\omega = \dot{\phi}$. We can deal with the constrained motion by introducing the Lagrange multiplier $\lambda(\dot{x} \tan \phi - \dot{y})$ to the Lagrangian. It will be useful to differentiate Eq. 3 both ways, giving

$$\ddot{y} = \ddot{x}\tan\phi + \dot{x}\dot{\phi}\sec^2\phi \tag{8}$$

$$\ddot{x} = \ddot{y}\cot\phi - \dot{y}\dot{\phi}\csc^2\phi \tag{9}$$

Using the Lagrangian in conjunction with the Langrange multiplier, we get two new equations of motion, the first being

$$m\ddot{y} = m\ddot{x}\tan\phi + m\dot{x}\dot{\phi}\sec^2\phi = \lambda. \tag{10}$$

The second now follows

$$m\ddot{x} = -\lambda\tan\phi + \frac{1}{2}mg \tag{11}$$

$$= -m\ddot{x}\tan^2\phi - m\dot{x}\dot{\phi}\tan\phi\sec^2\phi - \frac{1}{2}mg = 0 \tag{12}$$

Rearranging, and noting that $1+\tan^2\phi = \sec^2\phi$, we arrive at

$$\ddot{x} + \dot{x}\omega\tan(\omega t+\alpha) = \frac{g}{2}\cos^2(\omega t+\alpha) \tag{13}$$

This equation is a good sanity-check, since we expect the x direction to have a squared cosine dependence on gravity. With the hint given in the problem, the homogeneous solution is easy enough to guess,

$$x_{\mathrm{h}} = c_1\frac{\sin\omega t}{\omega} + c_2. \tag{14}$$

For the inhomogeneous, guess the right hand side divided by ω^2, where

$$x_{\mathrm{i}} = \frac{g}{2\omega^2}\cos^2(\omega t+\alpha) \tag{15}$$

$$\dot{x}_{\mathrm{i}} = -g\omega\cos(\omega t+\alpha)\sin(\omega t+\alpha) \tag{16}$$

$$\ddot{x}_{\mathrm{i}} = g\omega^2\left[\sin^2(\omega t+\alpha) - \cos^2(\omega t+\alpha)\right] \tag{17}$$

From inspection, the only correction we need is to multiply by $-1/2$ The full general solution for x is

$$x = c_1\frac{\sin\omega t+\alpha}{\omega} + c_2 - \frac{g}{4\omega^2}\cos^2(\omega t+\alpha). \tag{18}$$

The solution for y can be derived from the constraint equation

$$\dot{y} = c_1 \sin(\omega t + \alpha) + \frac{g}{2\omega}\sin^2(\omega t + \alpha) \tag{19}$$

$$= c_1 \sin(\omega t + \alpha) - \frac{g}{4\omega}\cos(2\omega t + 2\alpha) + \frac{g}{4\omega} \tag{20}$$

with solution

$$y = -\frac{c_1}{\omega}\cos(\omega t + \alpha) - \frac{g}{8\omega^2}\sin(2\omega t + 2\alpha) + \frac{gt}{4\omega^2} + c_3 \tag{21}$$

For part c, we can solve for the initial conditions, giving the equations of motion

$$x = -\frac{g}{4\omega^2}\left(\cos^2 \omega t - 1\right) = \frac{g}{8\omega^2}(1 - \cos 2\omega t), \tag{22}$$

$$y = \frac{gt}{4\omega} - \frac{g}{8\omega^2}\sin 2\omega t. \tag{23}$$

which is a cycloid.

One thought on “J98M.3”

Very good.
