J98M.3

Solution

Part(a)

The blade experiences no friction when it moves along itself or turns around its center. Since there is no component of normal force along rod, normal forces acts normal to the direction of motion which is at an angle ϕ with the y axis. The motion of the rod can be considered as the translation of center of mass and rotation about center of mass. Since there is no torque acting about the center of mass, we have $\ddot{\phi} = 0$.

From the free body diagram, equations of motion are :

$$m\ddot{x} = mg\sin\alpha - N\sin\phi \tag{1}$$

$$m\ddot{y} = N\cos\phi \tag{2}$$

$$I\ddot{\phi} = 0 \tag{3}$$

.

Part(b)

Since the blade cannot move translationally normal to itself, the velocity is always directed at an angle ϕ to the x-axis. Therefore the constraint can be written as

$$\dot{y} \ - \tan\phi\, \dot{x} = 0 \tag{4}$$

The time derivative of constraint gives us following expression for $\ddot{y}$

$$\ddot{y} = \sec^2\phi\dot{\phi}\dot{x} + \tan\phi\ddot{x} \tag{5}$$

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Part(c)

Using the expression for $\ddot{y}$ from the part(b),

$$m\ddot{x} = mg\sin\alpha - m\tan\phi[\sec^2\phi\dot{\phi}\dot{x} + \tan\phi\ddot{x}] \tag{6}$$

Since there is no torque acting on the rod, the angular velocity is conserved. Hence $\dot{\phi} = constant = \omega$.

Integrating we find that $\phi(t) = \omega t$ as $\phi_{t=0} \ = 0$.

$$\ddot{x} + \omega\tan(\omega t)\dot{x} - g\sin\alpha\cos^2(\omega t) = 0 \tag{7}$$

The above differential equation can be solved by the method of integrating factor. Substitute $\ddot{x} = \dot{u}$ and $\dot{x} = u$. Integrating factor is

$$e^{\int_0^t \omega\tan(\omega t)} = \frac{1}{\cos\omega t} \tag{8}$$

$$u(t) = \frac{\int_0^t g\sin\alpha\cos^2(\omega t) \times \frac{1}{\cos(\omega t)}\,\partial t}{1/\cos(\omega t)} = \frac{g\sin\alpha}{\omega}\sin(\omega t)\cos(\omega t) \tag{9}$$

Integrating $u(t)$ to get $x(t)$, constant of integration $x(0) = 0$,

$$x(t) = \frac{g\sin\alpha}{2\omega^2}\sin^2(\omega t) \tag{10}$$

$$\ddot{x} = g\sin\alpha\cos(2\omega t) \tag{11}$$

Putting this in equation of motion we get the normal force, N,

$$N = 2mg\sin\alpha\sin(\omega t) \tag{12}$$

It can be seen that the normal force is time dependent.

We will use the constraint again to get $y(t)$.

$$\dot{y} = \dot{x}\tan(\omega t) = u(t)\tan(\omega t) \tag{13}$$

$$y(t) = \frac{g\sin\alpha}{\omega}\int_0^t \sin(\omega t)\cos(\omega t)\tan(\omega t)\partial t \tag{14}$$

$$y(t) = \frac{g\sin\alpha}{2\omega}[t - \frac{\sin(2\omega t)}{2\omega}] \tag{15}$$

The angle $\alpha = 30^0$ so the trajectory can be described by

$$x(t) = \frac{g}{4\omega^2}\sin^2(\omega t) \tag{16}$$

$$y(t) = \frac{g}{4\omega}[t - \frac{\sin(2\omega t)}{2\omega}] \tag{17}$$

$$\phi(t) = \omega t \tag{18}$$

One thought on “J98M.3”

OK, very good, your solution is correct.