J98M.2

a) This problem lends itself well to center-of-mass coordinates. There are 4 degrees of freedom: x- and y-translation, rotation (θ) and spring elongation (l). The potential energy is due entirely to the spring, so $U = \frac{1}{2}k(l-d)^2$. Kinetic energy takes the form of translation, rotation, and stretching (measured from the center of rotation), so $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\frac{\dot{l}^2}{4}$. Simplifying, and taking $I = 2 \cdot M\left(\frac{l}{2}\right)^2 = M\frac{l^2}{2}$, the Lagrangian is:

$$\mathcal{L} = T - U = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2\frac{l^2}{2} + \frac{\dot{l}^2}{4} - \frac{k}{M}(l-d)^2\right)$$

Computing the Euler-Lagrange equation for l yields:

$$\frac{\partial \mathcal{L}}{\partial l} = \frac{1}{2}M\left(\dot{\theta}^2 l - \frac{2k}{M}(l-d)\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{l}} = \frac{1}{2}M\frac{\ddot{l}}{2}$$

Thus:

$$\ddot{l} = 2\left(l\dot{\theta}^2 - \frac{2k}{M}(l-d)\right)$$

Now, we need to eliminate $\dot{\theta}$, so we compute the Euler-Lagrange equation for θ:

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2}Ml^2\ddot{\theta}$$

Since $\frac{1}{2}l^2\ddot{\theta} = 0$, $\frac{1}{2}Ml^2\dot{\theta}$ is constant (which makes sense, because this is the angular momentum). After both masses start moving, the center of mass moves with velocity $v_0 = \frac{v_i}{2}$, and the masses rotate with $\omega_0 = \dot{\theta}_0 = v_i\frac{d}{2}$. Then $\frac{1}{2}Ml^2\dot{\theta} = Mv_i\frac{d}{2}$, so $\dot{\theta} = \frac{v_i d}{l^2}$.

Substituting this into the equation for $\ddot{l}$, we obtain:

$$\ddot{l} = \frac{2v_i^2 d^2}{l^3} - \frac{4k}{M}(l-d) \quad \blacksquare$$

b) Since the centrifugal force from rotation only ever acts outward, the spring will never be compressed (only stretched) once in motion. Thus, its shortest length is its rest length, d.

c) We know that when $\ddot{l} = 0$, the spring will be extended halfway, l_h (the acceleration is in the process of changing sign at that point). Then the full extension is $2l_h - d$ (so that we don't double-count the rest length). Setting the equation for $\ddot{l}$ equal to zero and rearranging yields:

$$\frac{v_i^2 d^2}{l_h^3} - \frac{2k}{M}(l_h - d)$$

If v_i is small, then l_h will be approximately d. Then we let l_h^3 go to d^3 , but keep $l_h - d$:

$$\frac{v_i^2}{d} = \frac{2k}{M}(l_h - d) \Rightarrow l_h = d + \frac{Mv_i^2}{2kd}$$

Thus, the full extension is $d + \frac{Mv_i^2}{kd}$.

One thought on “J98M.2”

a) Good, looks correct.

b) Fair enough. Can be shown analytically too.

c) Good, your answer is correct.

But I have one remark. Your assertion "We know that when $\ddot{l} = 0$, the spring will be extended halfway" doesn't look legitimate. How do we know this? It's true for a harmonic oscillator or any other oscillator with symmetric potentil. But actually it's not true in our case, because the effective potential in which the coordinate $l(t)$ propagates is not symmetric at all. However, since we are looking at the small v_i case, the motion is effectively harmonic, so that's why you are still getting a correct answer. I think, you should be more clear on that point. Or use different method, where you will analyze a point with $\dot{l} = 0$, not $\ddot{l} = 0$.