J98M.2

a)

This is a central force problem, therefore the motion of the system can be separated into center-of-mass motion and relative motion. Since the questions only ask about the relative motion, we ignore CM motion entirely.

Let $\vec{r}$ be the vector pointing from the left mass to the right mass, and represent it in the usual polar coordinates where $\theta = 0$ points right. (Notation: We will use $r(t)$ for the length of spring as function of time, and use the letter l for the angular momentum.) Then we write

$$L_{rel} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{1}$$

where $\mu = m/2$ is the reduced mass of the system. The initial conditions are

$$r(0) = d \tag{2}$$
$$\theta(0) = 0 \tag{3}$$
$$\dot{r}(0) = 0 \tag{4}$$
$$\dot{\theta}(0) = -\frac{v_i}{d} \tag{5}$$

and the potential term is

$$V(r) = \frac{1}{2}k(r-d)^2 \tag{6}$$

The Euler-Lagrange equation w.r.t. θ tell us the angular momentum is conserved. Using the initial conditions:

$$l = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = -\mu d v_i \tag{7}$$

The EL equation w.r.t. r gives, after substituting 6 7:

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{dV}{dr} = -\frac{\mu d^2 v_i^2}{r^3} - k(r-d) \tag{8}$$

b)

Energy of the relative motion

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\mu d^2 v_i^2}{r^2} + \frac{1}{2}k(r-d)^2 \\ &= \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r) \\ &\text{where } V_{eff}(r) = \frac{1}{2}\frac{\mu d^2 v_i^2}{r^2} + \frac{1}{2}k(r-d)^2 \end{aligned} \tag{9}$$

Since energy is conserved, and $\frac{1}{2}\mu\dot{r}^2$ attains minimum when $t = 0, \dot{r} = 0$: V_{eff} attains maximum at $t = 0, r = d$. At any t during motion, $V_{eff} \leq V_{eff}\big|_{t=0}$.

Since

$$\left.\frac{dV_{eff}}{dr}\right|_{r=d} = -\frac{\mu^2 v_i^2}{d} + 0 < 0 \tag{10}$$

r cannot decrease past $r(0) = d$ during the motion, i.e. the minimum length of the spring is d .

c)

Since $V_{eff}(r_{min}) = V_{eff}(r_{max})$:

$$\begin{aligned} \frac{1}{2}\frac{\mu d^2 v_i^2}{d^2} + \frac{1}{2}k(d-d)^2 &= \frac{1}{2}\frac{\mu d^2 v_i^2}{r_{max}^2} + \frac{1}{2}k(r_{max}-d)^2 \\ \implies \mu v_i^2\left(r_{max}^2 - d^2\right) &= k r_{max}^2 (r_{max}-d)^2 \end{aligned} \tag{11}$$

Since $r_{max} > d$, we can divide both sides by $(r_{max} - d$:

$$\mu v_i^2 (r_{max} + d) = k r_{max}^2 (r_{max} - d) \tag{12}$$

This is the equation satisfied by r_{max} .

One thought on "J98M.2"

Good solution.
