J98M.2

a) We define four generalized coordinates x, y, l, and θ, where x, y are the translation of the center of mass (COM) of the system, θ is the rotation around the COM, and l is the length of the spring.

The kinetic energy T of the system is given by:

$$T = \frac{1}{2}(I\omega^2 + mv^2) = \frac{1}{2} I\dot{\theta}^2 + M(\dot{x}^2 + \dot{y}^2 + \frac{\dot{l}^2}{4})$$

where $I = \frac{Ml^2}{2}$

and the potential energy U is:

$$U = \frac{1}{2} kx^2 = \frac{1}{2} k(l-d)^2$$

Resulting in:

$$L = T - U = \frac{1}{2}(I\omega^2 + mv^2) = \frac{1}{4} Ml^2 \dot{\theta}^2 + M(\dot{x}^2 + \dot{y}^2 + \frac{\dot{l}^2}{4}) - \frac{1}{2} k(l-d)^2$$

For the initial conditions:

We require momentum to be conserved when switching between frames. Initially, we are given a velocity on the left mass of v_i. To preserve momentum and initial energy, this is equivalent to giving a $v_i/2$ velocity to $\dot{y}$ and a v_i/d rotation rate to $\dot{\theta}$.

This results in: $l = d, \dot{y} = v_i/2, \dot{\theta} = v_i/d$

We take the Euler-Lagrange equations for l and θ:

θ: $\frac{Ml^2\dot{\theta}}{2} = constant$

Inserting our initial conditions, this becomes $l^2\dot{\theta}^2 = dv_i$

l: $\frac{d}{dt}(M\dot{l}/2) = \ \frac{1}{2}Ml\dot{\theta}^2 - k(l-d)$

and plugging in our previous expression:

$\ddot{l} = d^2 {v_i}^2 (\frac{1}{l^3}) - \frac{2k}{m}(l-d)$

b) When $\dot{l} = 0$, we have a min/max. Note that the initial conditions have this trait. Also note that the spring is at its resting position, so it is creating no force into compressing the spring. The rotation can only lengthen the spring, so we conclude that the minimum length of the spring is its resting length, d.

In fact, the other turning point is a maximum, which we determine in part c.

c) Assuming the system is symmetric across its inflection point, the system is halfway extended when $\ddot{l} = 0$. Consequently, we solve: $0 = d^2{v_i}^2(\frac{1}{l_0^3}) - \frac{2k}{m}(l_0 - d)$ for l_0 and define $l_{max} = 2l_0 - d$ For small v_i, $l_0 \to d$. Putting this into our previous expression, $l_0 = d + \frac{Mv_i^2}{2kd}$, where we only keep the difference between l_0 and d. Plugging in, we obtain $l_{max} = d + \frac{Mv_i^2}{kd}$

Another way of finding l_{max}:

We take and integrate equation of motion: $\ddot{l} = d^2{v_i}^2(\frac{1}{l^3}) - \frac{2k}{m}(l-d)$ from $t = 0$ to $t(l_{max})$, which is when l is at a maximum. The lefthand side is 0 so we only need to integrate the integral expression. At small v_i, l is approximately equal to d, and the first part of the expression becomes $2v^2/d$. The last part of the expression becomes $2k/m(l_{max} - d)$ and solving for l_{max} we recover our previous expression.

One thought on “J98M.2”

October 8, 2013 at 2:54 pm

You're on the right track.
Now just fix your solution.
Looks like you have typos in some formulas.
Also use \dot and \ddot for time derivatives to look like: $\dot{l}$ amd $\ddot{l}$.
Also, your solutions of part (b) and especially (c) do not look convincing enough (although the answers are correct). It would be nicer to use integrals of motion (energy and angular momentum) to derive the first order differential equation. And then to analyze that equation.