J98E.3

We set $R = a$

Laplace's equation: $\nabla^2 V = 0$

$$V(r,\theta) = \Sigma_{l=0}^{\infty}(A_l r^l + B_l/r^{l+1})P_l(cos(\theta))$$

For $r > R$:

Boundary condition: $r >> R$, $V = -E_0 rcos(\theta)$ so only A_1 survives (or $l = 1$) out of all the A_l's.

Boundary condition: $r = R$, $V = 0$.

Using these boundary conditions and the linear independence of P_l we obtain:

$$V(r > R, \theta) = (A_1 r + B_1/r^2)cos(\theta) = E_0 cos(\theta)(-r + R^3/r^2)$$

We find the electric field:

$$E(r > R, \theta) = -\nabla V = \ (E_0 + 2E_0 R^3/r^3)cos(\theta)\hat{r} + (-E_0 + E_0 R^3/r^3)sin(\theta)\hat{\theta}$$

At $r = R$:

$$E(R^+, \theta) = \ 3E_0 cos(\theta)\hat{r}$$

Inside the conductor $E = 0$:

$E(R^{-}, \theta) = 0$

We find the surface charge: $\sigma = \epsilon_0 (E_{out} - E_{in}) = 3\epsilon_0 E_0 cos(\theta)$

The surface charge generates an $E_{sigma} = \sigma/(2\epsilon_0) = (3/2) E_0 cos(\theta) \hat{r} = E_{ext}$on each side

So we may now find the differential force: $dF = E_{ext} dq = E_{ext} \sigma(\theta) dA = (9/2) \epsilon_0 E_0^2 cos^2(\theta) dA \hat{r}$

And so the force (on either top or bottom hemispheres):

(we replace $\hat{r}$ with $cos(\theta) \hat{z}$ because the x and y components of the force on one hemisphere cancel out due to symmetry around ϕ)

$$F = \int_0^{2\pi} \int_0^{\pi/2} d\theta d\phi R^2 sin(\theta)\ (9/2) \epsilon_0 E_0^2 cos^2(\theta) (cos(\theta) \hat{z})$$

$$F = 9\pi\ R^2\ \epsilon_0 E_0^2\ \int_0^{\pi/2} d\theta\ sin(\theta)\ cos^3(\theta)$$

$$F = -(9/4)\pi\ R^2\ \epsilon_0 E_0^2 cos^4(\theta)|_0^{\pi/2} = (9/4)\pi\ R^2 \epsilon_0 E_0^2$$

This force is required on each hemisphere to prevent them from separating.

One thought on “J98E.3”

December 8, 2013 at 11:37 pm

Everything looks correct.
Perfect, except that you didn't tell that $B_0 = 0$ was due to the vanishing total charge.
