J98E.2

a).

Take the fixed end as origin and let $\hat{r}$ point in the same direction as the rod. The dipole moment of the rod is

$$\vec{d} = \int_0^l \lambda r \hat{r} dr = \frac{\lambda l^2}{2} \hat{r} \tag{1}$$

The time-average power radiated by an oscillating electric dipole is

$$P = \frac{\omega^4}{12\pi\epsilon_0 c^3} \left|\vec{d}\right|^2 \tag{2}$$

and a rotating dipole, which radiates at constant power, can be thought as a superposition of linear oscillating dipoles. Therefore, for this question

$$\begin{aligned} P_E(t=0) &= 2\frac{\omega^4}{12\pi\epsilon_0 c^3}\left(\frac{\lambda l^2}{2}\right)^2 \\ &= \frac{\omega^4 \lambda^2 l^4}{24\pi\epsilon_0 c^3} \end{aligned} \tag{3}$$

b).

The moment of inertial of the rod relative to the origin is

$$I = \frac{ml^2}{3} \tag{4}$$

The kinetic energy is

$$T = \frac{1}{2} I\omega^2 \tag{5}$$

Take the time derivative of both sides, and assuming the only energy loss is from the radiated power in part (a):

$$\begin{aligned} -P_E = \dot{T} &= I\omega\dot{\omega} \\ \implies \dot{\omega} &= -\frac{P_E}{I\omega} \end{aligned} \tag{6}$$

Take time derivatives again:

$$\begin{aligned} -\dot{P}_E = \ddot{T} &= I\dot{\omega}^2 + I\omega\ddot{\omega} \\ \implies \ddot{\omega} &= \frac{-\dot{P}_E - I\dot{\omega}^2}{I\omega} \end{aligned} \tag{7}$$

Also, take the time derivative of 3:

$$\begin{aligned} \dot{P}_E &= \frac{\lambda^2 l^4}{24\pi\epsilon_0 c^3}\left(4\omega^3\dot{\omega}\right) \\ &= \frac{4P_E\dot{\omega}}{\omega} \\ &= -\frac{4P_E^2}{I\omega^2} \end{aligned} \tag{8}$$

Substitute 6 and 8 into 7 gives

$$\ddot{\omega} = \frac{3P_E^2}{I^2\omega^3} \tag{9}$$

The magnetic dipole of the rod is

$$\begin{aligned} \vec{m} &= \frac{1}{2}\int_0^l \vec{r}\times\vec{v}\lambda dr \\ &= \frac{\lambda\omega l^3}{6}\hat{z} \end{aligned} \tag{10}$$

and the power radiated by the magnetic dipole is

$$
\begin{aligned}
P_M &= \frac{\mu_0 \left|\ddot{\vec{m}}\right|^2}{6\pi c^3} \\
&= \frac{\mu_0}{6\pi c^3} \frac{\lambda^2 \ddot{\omega}^2 l^6}{36} \\
&= \frac{\mu_0}{6\pi c^3} \frac{\lambda^2 l^6}{36} \frac{9 P_E^4}{I^4 \omega^6} \\
&= \frac{\mu_0}{6\pi c^3} \frac{\lambda^2 l^6}{36} \frac{9}{\omega^6} \frac{\omega^{16} \lambda^8 l^{16}}{24^4 \pi^4 \epsilon_0^4 c^{12}} \frac{3^4}{m^4 l^8} \\
&= \frac{9 \times 3^4}{6 \times 36 \times 24^4 \pi^5} \frac{\mu_0^5 \lambda^{10} l^{14} \omega^{10}}{c^7 m^4} \\
&= \text{constant} \times \frac{\mu_0^5 \lambda^{10} l^{14} \omega^{10}}{c^7 m^4}
\end{aligned} \tag{11}
$$

One thought on “J98E.2”

December 8, 2013 at 11:07 pm

Good, looks correct.

Small note: a rotating dipole is itself a linear oscillating dipole (for which the formula is $P \propto |\ddot{\vec{d}}|^2$), or more precisely, it has a dipole moment which produces a radiation you computed.
