J98E.1 Electromagnetic Wave Incident on a Lossy Dielectric

Problem

A plane wave, $\vec{E} = E_0 exp(ikz - iwt)\hat{e}_x$, is incident on a lossy dielectric that fills the space for $z > 0$. (Figure included on copy of test). The dielectric is described by a real dielectric constant ϵ and conductivity σ. The space where $z < 0$ is a vacuum.

a) What is the dispersion relation, $k(\omega)$, in the dielectric?

b) At 2.5 GHz, what is the attenuation length of such a wave in a person? $(\mu = 1, \epsilon = 50, \sigma = 2 \times 10^{10} s^{-1})$

Solution

Begin with the Maxwell equations (Gaussian since that is how the values in part b are defined):

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right)$$

We can express the last two in terms of **E** and **H** like so:

$$\mathbf{J}_f = \sigma\mathbf{E} \text{ and } \mathbf{D} = \epsilon\mathbf{E} \rightarrow \nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi\sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\mathbf{B} = \mu\mathbf{H} \rightarrow \nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}$$

Take the partial derivatives with respect to time and make the substitution of $ik\times$ for $\nabla\times$.

$$ik \times \mathbf{H} = \frac{1}{c}\left(4\pi\sigma\mathbf{E} - i\omega\epsilon\mathbf{E}\right)$$

$$ik \times \mathbf{E} = \frac{1}{c} i\omega\mu\mathbf{H}$$

We can now take the cross product of the first equation and substitute in the second equation.

$$\frac{c}{\omega\mu} ik \times k \times \mathbf{E} = \frac{1}{c}\left(4\pi\sigma\mathbf{E} - i\omega\epsilon\mathbf{E}\right)$$

$$-k^2\mathbf{E} = -\frac{\omega\mu}{c^2}\left(i4\pi\sigma\mathbf{E} + \omega\epsilon\mathbf{E}\right)$$

$$k^2\mathbf{E} = \frac{\omega\mu}{c^2}\left(i4\pi\sigma\mathbf{E} + \omega\epsilon\mathbf{E}\right)$$

Solve for $k(\omega)$ with $\mu = 1$

$$k^2 = \frac{\omega^2}{c^2}\left(\epsilon + \frac{i4\pi\sigma}{\omega}\right)$$

$$\boxed{k = \frac{\omega}{c}\sqrt{\epsilon + \frac{i4\pi\sigma}{\omega}}}$$

We can see that we are going to get an attenuation factor from the imaginary part of k, because this will lead to a real, decaying term when k is plugged back into the plane wave equation. However, in its current form, k is not much use to us. We need to express it cleanly as a complex part and an imaginary part.

This is no problem if you have mathematica but we have to evaluate this by hand. I first tried to convert k^2 to a complex exponential and then take the root, but didn't have much luck. (That would still be the first thing I tried on the test.) Instead I used brute force:

$$k = \frac{\omega}{c}\left(a + ib\right)$$

where

$$(a+ib)^2 = \left(\epsilon + \frac{i4\pi\sigma}{\omega}\right)$$

$$(a^2-b^2) + 2iab = \epsilon + i\frac{4\pi\sigma}{\omega}$$

$$a^2 - b^2 = \epsilon$$

$$ab = \frac{2\pi\sigma}{\omega}$$

$$a = \frac{2\pi\sigma}{\omega b}$$

$$\left(\frac{2\pi\sigma}{\omega b}\right)^2 - b^2 = \epsilon$$

$$b^4 + \epsilon b^2 - \left(\frac{2\pi\sigma}{\omega}\right)^2 = 0$$

$$b^2 = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4 * \left(\frac{2\pi\sigma}{\omega}\right)^2}}{2}$$

Before we start plugging in stuff, we can go ahead and determine which root we will be using. We want an exponential decay in the region of $z > 0$. So we will want $b > 0$ so that ik results in $ia - b$. Also, at this point I started making some approximations to keep the arithmetic reasonable.

$$b^2 = -25 + \frac{1}{2}\sqrt{2500 + 1024\pi^2}$$

$$1024\pi^2 \approx 10240$$

$$\sqrt{12740} \approx 112$$

$$b^2 \approx -25 + \frac{112}{2} = 31$$

$$b \approx 5.5$$

$$a = \frac{2\pi\sigma}{\omega b} \approx 9$$

Therefore we can write k using $\frac{\omega}{c} \approx 8m^{-1}$ as:

$$k \approx 8m^{-1}\,(9 + 5.5i) = 72m^{-1} + (44i)m^{-1}$$

Plug in k:

$$\vec{E} = E_0 exp(ikz - iwt)\hat{e}_x = E_0 exp((-44m^{-1})z)exp((72i)z - i\omega t)$$

The attenuation term is $exp(-z/\lambda)$ where λ is the attenuation length, or skin depth.

$$\boxed{\lambda = \frac{1}{44m^{-1}} = 0.022m}$$

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