J98M.1—Hanging Rope

Problem

A piece of thin uniform unstretchable rope has length $2L$ and mass M. Its ends are attached to points at the same height separated by distance $2w$, and the rope hangs between them under the influence of gravity (of course, $w < L$). Let us set up coordinates (x, y) in the plane of the rope, so that the end points have equal values of y, and $x = \pm w$. You will be asked to determine the vertical coordinate of the rope, y, as a function of x.

a) Write down the functional of $y(x)$ that has to be minimized. What is the form of the constraint?

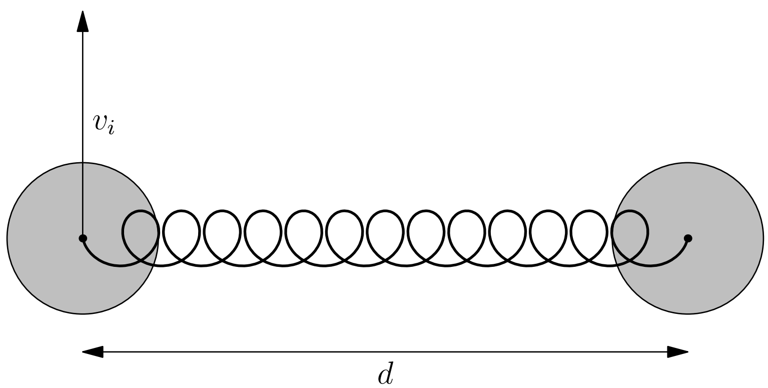
b) One may think of the functional to be minimized as an action for a 1-dimensional particle with coordinate y and time x. Find a conserved quantity.

c) For a given value of the conserved quantity, find $y(x)$. What is the equation relating the conserved quantity to w and L?

J98M.2—Pucks on a Spring

Problem

Two identical pucks of mass m can slide without friction on a horizontal table. Their centers are connected by an ideal massless spring of equilibrium length d and spring constant k. Initially the system is at rest. At $t = 0$ one of the pucks is hit sharply, which gives it velocity v_i normal to the spring.



a) Derive a differential equation for the length of the spring as a function of time, $l(t)$, for $t > 0$.

b) What is the minimum length of the spring during the motion?

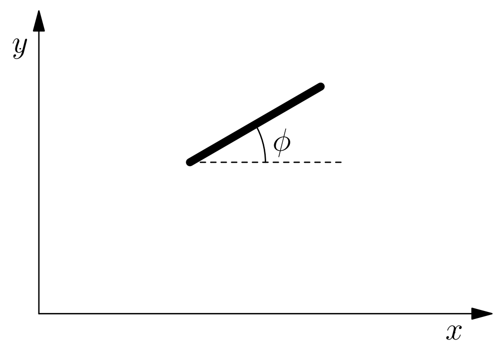
c) Derive an algebraic equation for the maximum length. Find its approximate solution for small v_i.

J98M.3—Ice Skate

Problem

As a simplified model for the motion of a skate, assume that the blade experiences no friction when it moves along itself and/or turns around its center. The blade cannot move translationally normal to itself.

Now consider a skate moving on an icy inclined plane which makes a 30 degree angle with the horizontal. In view of the assumption above, you may think of the blade as a thin uniform rod of mass M moving on the plane under the influence of gravity subject to the constraint that it cannot move translationally normal to itself. Introduce Cartesian coordinates x and y on the plane, with x pointing down the incline. The blade is characterized by its center of mass position (x, y), and the angle ϕ it makes with the x-axis.



a) Write down the equations of motion including the reaction force normal to the blade.

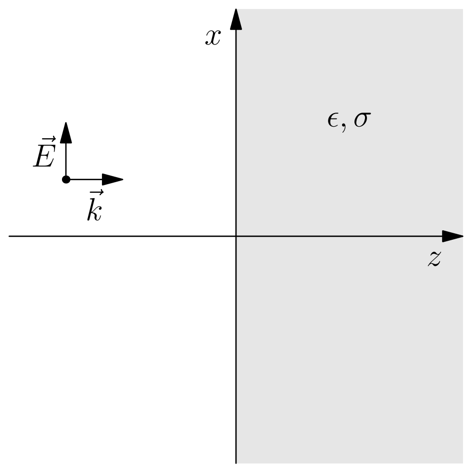
b) Write down the constraint on the motion in terms of x, y, ϕ, and their time derivatives.

c) At time $t = 0, x = y = \phi = \dot{x} = \dot{y} = 0$ and $\dot{\phi} = \omega$. Find the subsequent trajectory. Hint: The reaction force normal to the blade is proportional to $\sin(\omega t)$.

J98E.1—Electromagnetic Wave Incident on a Lossy Dielectric

Problem

A palne wave, $\vec{E} = E_0 \exp(ikz - i\omega t)\hat{e}_x$, is incident on a lossy dielectric that fills the space for $z > 0$ as shown in the figure. The dielectric is described by a real dielectric constant ϵ and conductivity σ. The space where $z < 0$ is a vacuum.



a) What is the dispersion relation, $k(\omega)$, in the dielectric?

b) At 2.5 GHz, what is the attenuation length of such a wave in a person? ($\mu = 1, \epsilon = 50, \sigma = 2 \times 10^{10}$ s^{-1}).

J98E.2—Rotating Charged Rod

Problem

A thin uniform rod of length l and mass M has constant linear charge density λ. Its endpoint is rigidly attached to a vertical axis at right angles. The rod is given angular velocity $\omega \ll c/l$ about the axis at $t = 0$. You may assume that the electrostatic energy stored in the rod is much smaller than the kinetic energy of the rod. No external torques are applied for $t \geq 0$.

a) What is the power radiated at $t = 0$ due to the electric dipole emission?

b) Estimate, up to a dimensionless constant of proportionality, the power radiated at $t = 0$ due to the magnetic dipole emission.

J98E.3—Conducting Hemispheres

Problem

An insulated, uncharged, conducting, spherical shell of radius a is placed in a uniform electric field of magnitude E_0. Suppose the shell is cut into two hemispheres at its equator (in the plane perpendicular to the field). What force is required to keep the hemispheres from separating?

J98Q.1—Parity Violation

Problem

A spin $\frac{1}{2}$ particle of mass m moves in a spherical harmonic oscillator potential and is also subject to a parity violating perturbation. The Hamiltonian is $H = H_0 + H_1$, with

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 \quad \text{and} \quad H_1 = \lambda\vec{\sigma}\cdot\vec{r},$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices.

As a measure of the parity violation, compute the expectation value $\langle z\,\sigma_z\rangle$ for the ground state, to first order in λ.

J98Q.2—Two Delta-function Potential

Problem

A particle of mass m moves in one dimension subject to the potential

$$V(x) = -a\delta(x + L) + a\delta(x - L),\ \ a > 0.$$

The system has one bound state.

a) Derive a transcendental equation relating the energy of the bound state to m, a, and L.

b) Calculate the energy of the bound state to leading order for small L, and sketch the wave function.

J98Q.3—Hyperfine Structure

Problem

The hyperfine structure of the $n = 1$ level of hydrogen arises from a coupling between the electron and proton spins of the form

$$H_{hyperfine} = a\vec{s}_e \cdot \vec{s}_p,$$

where a is a positive constant. The other terms in the hydrogen atom Hamiltonian do not lift the degeneracy of the $n = 1$ level and may be ignored in this problem.

a) find the energies and degeneracies of the $n = 1$ hyperfine levels.

A uniform magnetic field $\vec{B}$ is switched on for a period of time. For simplicity, assume that the field is constant for $0 < t < T$ and zero at all other times.

b) To a very good approximation we can ignore the magnetic interaction of the proton with the field compared to that of the electron. Briefly explain why.

c) Given that the atom was in its ground state before the magnetic field was turned on, what is the probability that it is in its ground state for $t > T$?

J98T.1—Quark-Antiquark Pairs

Problem

In this problem we will use statistical mechanics to obtain a crude estimate of the number of quark-antiquark pairs produced in a high energy collision between hadrons. We begin by assuming that the initial consequence of the collision is to distribute the incoming energy E in a ball of radius R which is comparable to the Compton wavelength of the pion. This energy is then assumed to go into producing an equilibrium gas of quark-antiquark pairs whose number we wish to estimate. The quarks and antiquarks are fermions whose masses may be ignored in theis process.

a) Evidently, the number of quarks or antiquarks is not conserved. What is the average level occupation, $f(\epsilon)$, per species of quark/antiquark at energy ϵ?

b) Neglecting the masses of the quarks, what is the density of states per unit volume per unit energy at energy ϵ, per species of quark/antiquark?

c) The number of quark species is 2 (spin) $\times$ 3 (color) $\times$ 3 (light flavors) $= 18$. Calculate the equilibrium temperature of the quark-antiquark gas in terms of E and R.

d) Hence determine the number of quark-antiquark pairs in equilibrium in terms of E and R.

Possibly useful integrals:

$$\int_0^\infty dx \frac{x}{e^x+1} = \frac{\pi^2}{12}$$
$$\int_0^\infty dx \frac{x^2}{e^x+1} = \frac{3}{2}\zeta(2) \approx 1.8$$
$$\int_0^\infty dx \frac{x^3}{e^x+1} = \frac{7\pi^4}{120}$$

J98T.2—Cooling Liquid Helium

Problem

Consider a closed dewar containing liquid ^{4}He (whose atoms are spin zero bosons for our purposes) in equilibrium with its vapor at low temperatures.

a) The latent heat of vaporization per atom of ^{4}He is l at $T = 0$ which fixes the chemical potential. What is the vapor pressure at temperatures $k_B T \ll l$? You may neglect the temperature dependence of the chemical potential and make other reasonable approximations.

b) ^{4}HE at one atmosphere of pressure boils at about 4K. Use your result from part a) to get a *rough* estimate of l based on this datum.

c) The dewar is not perfectly insulating whence heat leaks into the liquid ^{4}He at a rate $\dot{Q}$. At what rate $\dot{V}$ (volume per unit time) doe a pump have to remove the vapor to keep the (low) temperature from rising? (Pumping is a simple means of cooling liquid ^{4}He.)

Useful numbers:

$$m_{\text{He}} \approx (2/3) \times 10^{-23} \text{ g}$$
$$k_B = 1.3807 \times 10^{-23} \text{ J/K}$$
$$h = 6.6262 \times 10^{-34} \text{J s}$$

J98T.3—Spins in a Magnetic Field

Problem

A sample, comprised of N independent spins ($s = 1/2$), sits in an external magnetic field $\boldsymbol{H}$. Its Hamiltonian is given by

$$H = -g\mu_B \sum_n \boldsymbol{s}_n \cdot \boldsymbol{H},$$

where $g = 2$ and μ_B is the Bohr magneton.

a) Write down the partition function Z.

b) Calculate the sample's entropy $S(H, T)$, and make a rough sketch of S versus the temperature T for a fixed field H_1. Show that S is a function of only one quantity x instead of two (T and H). How is x related to T and H? What is the T dependence of S in the low temperature limit?

c) The sample is initially connected to a heat bath at temperature T_0, with the field at H_1. The external field is increased slowly from H_1 to H_2 in an isothermal process. Calculate the heat Q exchanged with the bath. Which way does the heat flow? (Sketch the curve for S vs. T for a larger field.)

d) When the field reaches H_2, the link to the heat bath is removed. The field is then slowly reduced back to H_1 in an adiabatic process. Calculate the final temperature T_f of the sample (Hint: The actual form of S in part b) is not needed here. You may use a simple scaling argument.)