Department of Physics, Princeton University

Graduate Preliminary Examination
Part I

Thursday, January 12, 2023
9:00 am - Noon

Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet.

Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Mechanics

1. **Elliptical orbits around Sagittarius A***

   There appears to be a black hole at the center of our galaxy, Sagittarius A* with a mass M_b of about 4×10^6 solar masses. Astronomers have been able to determine the mass M_b of the black hole by measuring the period T and the major axis a of the highly elliptical orbits of stars with mass $m_s << M_b$ which orbit the black hole. Using classical Newtonian gravity, *derive* an expression for the mass M_b of Sagittarius A* from T, a of the star orbit and G, the gravitational constant.

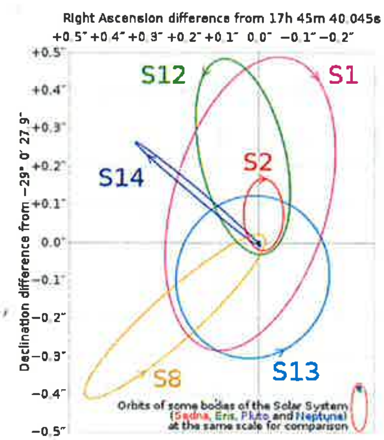


Figure 1: Representative orbits of some stars orbiting Sagittarius A*.

2. **A hanging string**

Suppose we have a string of length L and linear mass density λ (a constant). Suppose we fix the ends at points (x_i, y_i) and (x_f, y_f).

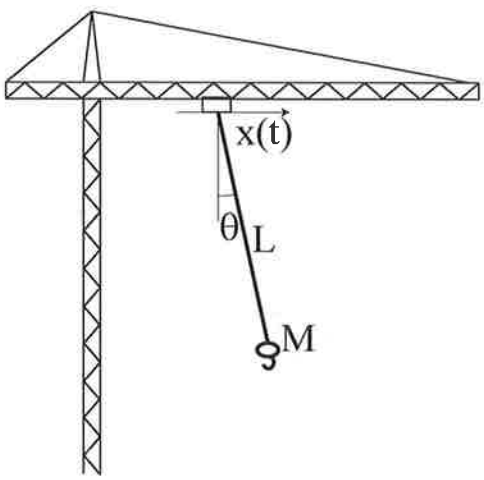
(a) Suppose you put your finger on the hanging string and push down so it forms a V-shape. Does the center of mass of the string rise or lower as you push down? This can give you a hint on how to attack the next part of this problem.

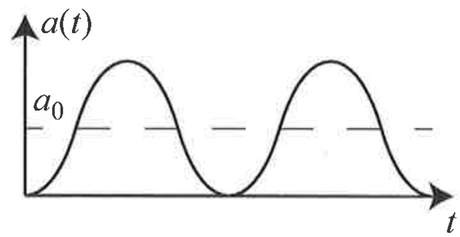
(b) Derive the shape $y(x)$ the string makes, hanging under gravity g. You do not need to explicitly fix the integration constants in your answer, but explain in principle how you would fix them.

(c) You don't need the answer to (b) to answer this. The solution $y(x)$ for the hanging string when inverted so the curve is up is often seen in bridges and other load bearing structures. Why?

3. **Crane with trolley**

A construction crane installed near Jadwin has a heavy hook mounted from a horizontally-moving trolley, as shown in the figure. Consider the hook as a simple pendulum of mass M and length L hanging from the trolley. The position of the trolley $x(t)$ is controlled by the operator of the crane.





The trolley and hook are initially at rest and the hook is hanging straight down. It is desired to start moving the trolley to the right with a constant velocity without any oscillations in the pendulum after the constant velocity is reached. The acceleration of the trolley as a function of time is shown on the right. It can be described mathematically as $\ddot{x} = a(t) = a_0(1 - \cos(\omega t))$ for $0 < t < 4\pi/\omega$ and $\ddot{x} = 0$ for $t > 4\pi/\omega$ and $t < 0$.

a) Derive an equation of motion for $\theta(t)$ for an arbitrary $x(t)$. You can assume that $\theta(t)$ always remains small.

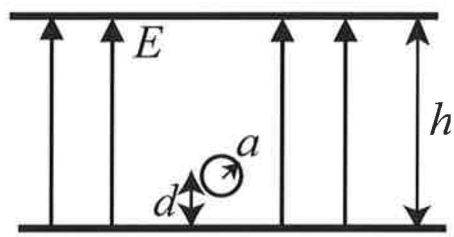
b) Considering the resulting equation as a driven simple harmonic oscillator, show that it is possible to pick ω so that the pendulum does not oscillate after the platform is moving with a constant velocity.

c) Assuming that one picks the highest possible ω (shortest possible time $t_f = 4\pi/\omega$), what is the final velocity of the platform?

Section B. Electricity and Magnetism

1. **Dielectric ball inside capacitor**

A small neutral dielectric sphere with radius a and relative permeability ϵ_r is placed between two perfectly conducting metal plates of a parallel plate capacitor. The sphere is located a distance d from the closest plate. Consider the limit when $a \ll d \ll h$. We want to find the force acting on the neutral sphere when the electric field inside the capacitor is equal to E.



a) Calculate the size of the electric dipole moment induced in the dielectric sphere due to the presence of a (nearly) uniform electric field E.

b) Consider the interaction of the dipole moment with the lower conductive plate of the capacitor which will make the electric field not perfectly uniform and generate a force on the sphere. Find the magnitude and direction of the force.

2. **Plasma Waves**

A "tenuous plasma" consists of free electrons with mass m and charge e. There are n electrons per unit volume. Assume that the electron density is uniform and that interactions between the electrons may be neglected. Electromagnetic waves (frequency ω and wave number k) are incident on the plasma.

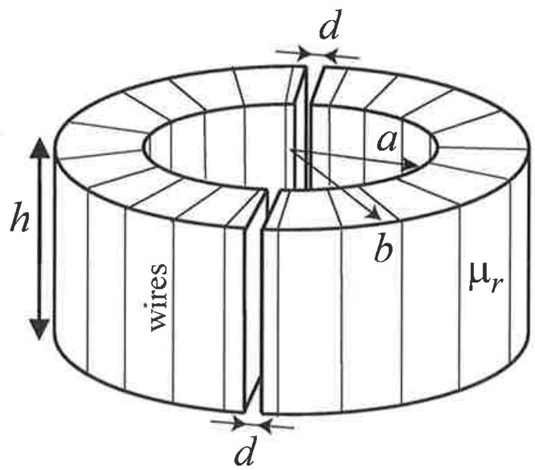
(a) Find the conductivity σ of the plasma as a function of ω.

(b) Find the dispersion relation; i.e. find the relation between k and ω.

(c) Find the index of refraction as a function of ω. What does it tell you about the speed of wave propagation in the plasma? The plasma frequency is defined as $\omega_p^2 = ne^2/m\epsilon_0$ (in SI units). What happens if $\omega < \omega_p$?

3. **Gapped Toroid**

Inductors used in electronic circuits often have the geometry of a toroidal coil with a magnetic core cut in half, as shown in the figure:



The magnetic core ring is made from a linear magnetic material with relative permeability μ_r that is very large, $\mu_r > 1000$. The core ring has inner radius a, outer radius b and height h, these three dimensions are of similar size. The ring is cut in half and the two halves are pulled apart by a very small distance d, with $d \ll a, b, h$. A toroidal coil is wound around the ring with total number of turns N.

(a) First consider the case when there is no gap, $d = 0$. Calculate the inductance of the toroidal coil wound around the core.

(b) Now include the effect of a small gap, $0 < d \ll a, b, h$ and calculate the inductance of the coil.

(c) Find a simplified expression for the inductance valid in the regime when $\mu_r d \gg a, b, h$ but $d \ll a, b, h$

(d) What might be a practical reason why gapped toroids are used often instead of non-gapped ones?

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Graduate Preliminary Examination
Part II

Friday, January 13, 2023
9:00 am - Noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

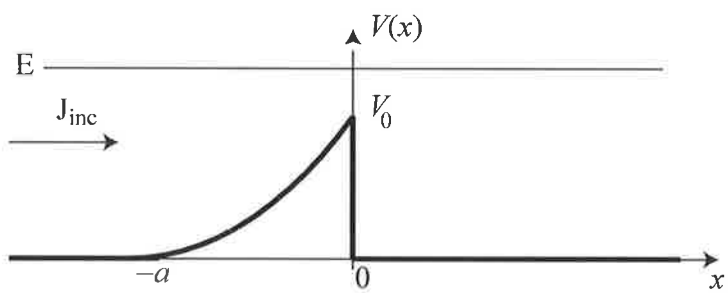
Section A. Quantum Mechanics

1. **Reflection from a skateboard ramp**

Consider a one-dimensional potential given by ($a > 0$)

$$V(x) = \begin{cases} 0 & x < -a \\ \dfrac{(x+a)^2}{a^2} V_0 & -a < x < 0 \\ 0 & x > 0 \end{cases}$$

A particle with mass m and energy E is propagating from the left, its energy is $E > V_0$.



(a) Write a wavefunction for the particle for $x < 0$ within the WKB approximation.

(b) Calculate the particle flux given by the wavefunction from part (a) and show that it is a constant within WKB approximation.

(c) Calculate the reflection coefficient R for particle's reflection from the ramp.

(d) Consider the limit of $a \to 0$ while V_0 remains constant. What happens to the reflection coefficient according to the WKB approximation? Is it accurate or not?

2. **Transitions in a 1-D potential**

Consider a particle of mass m in a 1-D infinite square potential well: $V(x) = 0$ for $|x| < a$ and $V(x) \to \infty$ for $|x| > a$. The particle is initially in the ground state of the potential. You would like to transfer the particle to the *second* excited state of the potential with probability close to 100%.

To perform this excitation you can turn on a pulse of a perturbation potential $V_1(x,t) = \lambda x \cos(\omega t)$. You can control the frequency of the oscillation ω and the duration of the pulse during which the perturbation potential is turned on, it's either on or off. The strength of the perturbation λ is fixed and is very weak, so that it would take many oscillations at frequency ω to excite the particle.

(a) Write down the energies and wavefunctions of the particle in the ground state, the first excited state and the second excited state.

(b) Describe a strategy for exciting the particle from the ground to the second excited state with probability of essentially 100%, within time-dependent perturbation theory. You may restrict to first order perturbation theory. Note that to achieve the required transition you are allowed to use a sequence of pulses.

(c) For the method you identified, describe the frequencies and pulse durations for the perturbation potential that you would use. It is sufficient to state an equation for calculating the pulse duration without evaluating the integrals involved.

(d) Now consider the same situation, except that the particle is in a 1-D harmonic oscillator potential $V(x) = kx^2/2$ instead of an infinite square well. Explain why it is much harder to transfer the particle from the ground state to the second excited state of the potential with 100% probability.

3. **Charged particle in perpendicular E and B fields**

A spinless charged particle of charge q moves in the xy-plane under the influence of a uniform magnetic field in the z-direction, $\vec{B} = (0, 0, B)$ (with B a non-zero constant), and a uniform electric field in the x-direction, $\vec{E} = (E, 0, 0)$ (with E a non-zero constant). In this problem we can ignore the motion in the z-direction: the particle is confined to move on the xy-plane. Please work in the Landau gauge $\vec{A} = (0, xB, 0)$ for the vector potential.

(a) Write down the Hamiltonian for the particle.

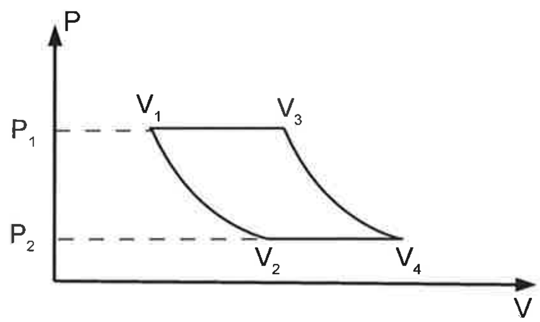
(b) Identify an observable that commutes with the Hamiltonian, and write a corresponding suitable ansatz for the wavefunction of the energy eigenstate.

(c) Find the energy eigenstates of the system (you do not need to explicitly fix the normalization of the eigenfunctions).

(d) Based on the result you found in part (c), are the energy eigenstates degenerate or non-degenerate? Does the situation change when the electric field $E = 0$?

Section B. Statistical Mechanics and Thermodynamics

1. **A heat engine**



A heat engine operates by taking its working fluid quasistatically through the cycle shown in the figure. Two segments of the cycle are at constant pressure, as shown, while the other two segments are adiabatic. The cycle goes clockwise (1 to 3 to 4 to 2 to 1) to make a heat engine. The energy of the working fluid at the vertices of the cycle is U_1, U_2, U_3, U_4, and the volumes are indicated in figure. Note that the points are not numbered in order going around the cycle. Be careful to use the same numbering as in the figure.

(a) First consider a general fluid (not an ideal gas). For each of the four segments of the cycle, give Q, the heat that enters the fluid during that segment (Q is negative if heat leaves the fluid) in terms of the given pressures, volumes, and energies. For the constant pressure segments of the cycle, write Q for those segments in terms of the appropriate free energy.

(b) What is the efficiency of this heat engine if it is operated reversibly? Express the result in terms of the enthalpies H_1, H_2, H_3, H_4 of the four vertices of the cycle.

(c) Next (and **not** earlier in this problem) assume the working fluid is a 3D classical ideal gas of diatomic molecules, and the temperature is such that the molecular vibrations are not excited, while the molecular rotations are highly excited. Take P_1, P_2, V_1 and V_3 as the given parameters. Give U_1, V_2, and the efficiency in terms of these given parameters.

2. **An ideal gas in two dimensions**

Consider point particles of mass m with no internal degrees of freedom that are confined in a two dimensional box of size $L \times L$ with area $A = L^2$. The particles move nonrelativistically only in two dimensions.

(a) Derive the partition function Z_1 of one of such particles in this box at the temperature T. First write Z_1 as a sum over the appropriate quantum numbers of the particle, then state the condition under which the sum can be approximated by an integral, and finally compute the integral explicitly. For the remainder of this problem, you can assume we are in the regime where sums can be approximated by integrals.

(b) Now consider an ideal gas of a large number N of indistinguishable and non-interacting particles in this box. Derive the partition function Z_N of the N particles in terms of Z_1 in the classical regime. Use this to derive the Helmholtz free energy F of this gas in the classical regime. You may use the Stirling approximation $N! \approx N \log N - N$.

(c) What is the entropy S of this classical gas? What is the internal energy U?

(d) If the box is expanded or compressed reversibly with the gas at constant entropy, starting from area A_0 and temperature T_0, what is the temperature as a function of area $T(A)$?

(e) If instead these particles are spin-1/2 fermions in two dimensions, what is the Fermi energy $\epsilon_F(N, A)$ of this two-dimensional Fermi gas?

3. **Non-interacting spins in a magnetic field**

Consider a system of N non-interacting quantum spins with spin 1/2 in an external uniform magnetic field B along the z direction. The spins are distinguishable (e.g. by placing them at different fixed spatial positions). The Hamiltonian is

$$H = -\sum_{i=1}^{N} \mu B \sigma_i^z$$

where μ is the magnetic moment and σ_i^z the Pauli matrices for each spin. The spins are in equilibrium at temperature T.

(a) Find the partition function Z and the free energy F of the system.

(b) Compute the magnetic susceptibility χ of this spin system. It is defined by $\chi = \frac{\partial M}{\partial B}$, where M is the average magnetization $M = \langle \hat{M} \rangle$, with $\hat{M} = \sum_i \mu \sigma_i^z$.

(c) Consider the variance of the magnetization

$$(\delta M)^2 = \langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2$$

Show that the variance is related to the magnetic susceptibility by $(\delta M)^2 = k_B T \chi$.

(d) Now consider a system of N *classical* spins, with the Hamiltonian of the same form as above

$$H = -\sum_{i=1}^{N} \vec{\mu}_i \cdot \vec{B}$$

The magnetic moments $\vec{\mu}_i$ have fixed magnitude μ and can point in any direction on the unit sphere. For such classical spin system, repeat the calculation of the partition function Z, free energy F and magnetic susceptibility χ.

(e) Show that in the large temperature limit, the magnetic susceptibility for both the quantum and classical spins takes the Curie's law form $\chi \approx \frac{C}{T}$, with C a constant. Determine the value of C for the quantum and classical spins.