Department of Physics, Princeton University

Graduate Preliminary Examination

Part I

Thursday, January 6, 2022
9:00 am - 12:20 noon
plus one hour for uploading the answers

Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet.

Be sure to label each booklet with your name, the section title and the problem number (e.g. E&M 2), and upload the booklets into the corresponding Canvas bins.

Do not forget to upload also your honor code pledge (Q7), and to hit the “submit” button when all is done.

Once the exam is submitted it cannot be reopened.
Section A. Mechanics

1. A billiard ball of radius $R$, mass $m$ and rotational inertia $I$, is at rest on a table at a point $A$. As shown in the figure, it is struck with a cue stick at a point $d$ below its center of mass CM ($d < R$).

The velocity of the CM immediately after impact is $v_0$. The ball slides and rotates before it pauses instantaneously at a spot $B$, a distance $X$ from $A$ (see figure). Then the linear velocity of the CM reverses sign. On the return path, the ball slides and rotates over a short distance up to a point $C$ at which it starts rolling without slipping. The coefficient of kinetic friction between the ball and table is $\mu_k$.

\[ \text{d} \quad v_0 \quad \text{R} \quad \text{CA} \quad \text{B} \quad \text{X} \]

(a) Find the impulse $J$ and the angular velocity $\omega_0$ (about the CM) immediately after impact?

(b) What are the forces and torques acting on the ball during the slide-and-rotate segment ($A \to B \to C$), and during the rolling without slipping segment ($C \to A$)?

(c) Find the distance $X$ (between $A$ and $B$).

(d) Find the velocity of the center of mass when it rolls past the original spot $A$. 
2. A bead is free to slide without friction on a vertical loop which is rotated at angular velocity $\omega$ around its vertically oriented diameter.

(a) Find the values of the polar angle $\theta$ at which the bead may be stationary (with the angle defined so that $\theta = \pi$ corresponds to the loop’s lowest point).

*Hint: the answer can be derived through either Newtonian force analysis, or the Lagrangian formulation.*

(b) Show that for $\omega$ large enough (i.e. $\omega > \omega_0$) the bead’s position $\theta = \pi$ is unstable. Calculate the critical value $\omega_0$, and the bead’s stable position for $\omega > \omega_0$.

(c) Describe the bead’s dynamics in the vicinity of the stable orbit.
3. Consider transversal waves in a taut string of linear mass density $\mu$ and tension $T$. At one end, $x = 0$, the string is attached to a ring of mass $m$ that slides freely (no friction) on a vertical pole, and at the other, $x = L$, it is fixed at height $y = 0$. Ignore gravity (assuming its effect is much smaller than that of the tension).

(a) Write the differential equation satisfied by the displacement $y(x, t)$ with oscillations of low amplitude.

(b) State explicitly the relevant boundary conditions by which the equation is to be supplemented at $x = 0$ and $x = L$.

(c) Identify the time-periodic solution (i.e. standing wave) of the lowest wavenumber.
Section B. Electricity and Magnetism

1. Consider a very long solenoid of radius $a$ that carries current $I_1(t)$, which decays linearly in time. The solenoid is concentric with a large circular loop (loop 2) of radius $b$ ($b \gg a$). The loop is comprised of a single wire of total resistance $R$. Due to the decay of $I_1(t)$ the field inside the solenoid decreases as

$$B(t) = B_0 - \alpha t.$$ 

As a result, a current $I_2$ is induced in the loop 2.

(a) Find the induced current $I_2$ (in terms of the parameters mentioned).

(b) Compare the power dissipated in loop 2 with the power radiated from the solenoid, as computed through the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ integrated over the cylindrical surface enclosing the solenoid.

*Hint:* On that surface, where do $\mathbf{E}$ and $\mathbf{B}$ come from?
2. **Electricity and Magnetism**

A very long wire with axis parallel to \( \hat{x} \) carries a current \( I \). It is at rest in the lab frame. A positive charge \( q \) with velocity \( v \parallel \hat{x} \), in direction parallel to the wire’s axis, with the same orientation as \( I \), at a radial distance \( r \).

![Diagram of a long wire with a charge moving parallel to the wire]

(a) Compute the magnetic field \( \vec{B} \) due to the current at the location of the moving charge, and the resulting Lorentz force \( \vec{F} \) on the charge \( q \).

(b) In the rest frame of the moving charge the picture is quite different: the charges velocity vanishes and hence so does the Lorentz force on it. However, force equal to \( \vec{F} \) of (a) may appear due to the electric field of a uniformly spread charge, along the long the wire, of linear charge density \( \rho_0 \). Compute that density.

The next calculation aims at testing whether the above two perspectives can be reconciled through the relativistic Lorentz contraction formula.

(c) Assume, for simplicity, that in the lab frame the electric charges in the wire consist of static positive charges of linear charge density \( \lambda \) and negative charges, of linear charge density \( -\lambda \), which move at velocity \( u \) (with \( u \cdot v < 0 \)). (That is: in the lab frame the total charge density vanishes, but the charges move at different velocities.) Express \( u \) in terms of the electrical current \( I \).

(d) With input from special relativity, calculate the corresponding combined charge density \( \lambda' \) as it appears in the reference frame of the moving particle (at \( v \ll c \)). Does it agree, to the leadings order, with the charge density computed in (b)?

*Hint:* Recall i) the Lorentz contraction formula \( L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = L_0 / \gamma \)

ii) the relativistic velocity addition rule \( u' = \frac{u - v}{1 - uv/c^2} \) where \( u \) is the velocity of an object measured in the lab, \( u' \) its velocity measured in the rest frame of an observer moving at velocity \( v \), and \( c \) the speed light.
Within a solid block of aluminum, a cavity is created in the shape of a torus with minor radius $a$ and major radius $R$ (see left figure). In the horizontal plane of the torus, a slot is cut out of height $d$, so that the cross section of the cavity is comprised of two circular holes of radius $a$ with centers separated by $2R$ and connected by a narrow slit of width $d$ (the slot). The cavity has cylindrical (azimuthal) symmetry about the symmetry axis (dashed line).

Using a very small antenna (not drawn), we can excite oscillating electromagnetic fields inside the cavity as a resonant mode. Within the metal, the fields and currents are confined to a thin layer of skin depth $\delta$. For simplicity assume that $R \gg a \gg d \gg \delta$.

The equivalent $LC$ circuit is shown on the right.

(a) Make a rough sketch of the $E$ and $B$ field lines in the cavity.

Hints: It may be helpful to remember the boundary conditions. At the surface of the metal, $B$ is tangential to the surface, whereas the tangential component of $E$ is negligible compared with its normal component.

Which part of the cavity would you identify with the equivalent circuit’s capacitor?

(b) The symmetry of the problem suggests two closed loops $\Gamma_1$ and $\Gamma_2$ to which we can fruitfully apply the curl theorem:

$$\int_{S_i} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\Gamma_i} \mathbf{F} \cdot d\mathbf{\ell}, \quad (i = 1, 2)$$

where $\mathbf{F}$ is either $\mathbf{E}$ or $\mathbf{B}$.

Identify the loops $\Gamma_1$ and $\Gamma_2$ and write down two equations that relate the $E$ and $B$ fields and their derivatives with respect to time.

Hint: Focus on the magnetic flux and electric flux and their derivatives with respect to time.

(c) From the two equations, find the resonance frequency of the cavity.

(d) For the equivalent circuit shown, calculate the equivalent capacitance $C$ and inductance $L$. 

Department of Physics, Princeton University

Graduate Preliminary Examination
Part II

Friday, January 7, 2022
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.
Section A. Quantum Mechanics

1. An electron is at rest in a tilted magnetic field \( \mathbf{B} = B(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) where \( \theta \) and \( \phi \) are the polar and azimuthal angles relative to lab axes \( x, y, z \) (see figure).

The Hamiltonian is given by

\[
\mathcal{H} = -\mu_B \mathbf{\sigma} \cdot \mathbf{B}.
\]  

where \( \mu_B \) is the Bohr magneton and \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(a) Find the eigenenergies \( E_+ \) and \( E_- \) of \( \mathcal{H} \) (in terms of \( B, \theta \) and \( \phi \) and other constants)

(b) Find the (normalized) eigenfunctions, expressed in terms of the eigenstates of \( \sigma_z \)

(c) You perform a measurement of the spin at time \( t = 0 \) and find that it is polarized along \( \hat{z} \), i.e. the state at \( t = 0 \) is

\[
\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Write down the wave function \( \psi(t) \) at subsequent times \( t > 0 \).
2. Consider a pair of quantum spins $\vec{S}_j = (S_j^x, S_j^y, S_j^z)$, $j = 1, 2$ of magnitude $S = 1/2$, interacting through the Hamiltonian

$$H_J^{(2)} = J \vec{S}_1 \cdot \vec{S}_2.$$  

(a) The operator $H_J$ can be equivalently expressed in terms of the magnitude of the total spin operator triplet $\vec{S}_T = \sum_{j=1}^{2} \vec{S}_j$. State this relation.

(b) What are the energies and degeneracies of ground states of $H_J$ for the cases:
   i) $J = -1$ (ferromagnetic)
   ii) $J = +1$ (antiferromagnetic).

(c) Next, extend the ground state energy calculation to $N$ quantum spins of magnitude $S = 1/2$, with the Hamiltonian

$$H_J^{(N)} = J \sum_{1 \leq i < j \leq N} \vec{S}_i \cdot \vec{S}_j$$

That is: calculate the ground states energies of $H_J^{(N)}$ for:
   i) $J = -1$ (ferromagnetic)
   ii) $J = +1$ (antiferromagnetic).

*Hint: in one of these cases one needs to pay attention to the parity of $N$.***
3. A particle of mass $m$ moves on a line under an attractive delta function potential, with the Hamiltonian

$$H_\lambda = \frac{1}{2m} p^2 - \lambda \delta(x)$$

(a) Present a variational argument proving that for any $\lambda > 0$ the operator $H_\lambda$ has at least one eigenstate of negative energy.

Hint: it suffices to consider test functions of the form $\psi_R(x) = \frac{1}{\sqrt{R}} \phi(x/R)$.
E.g. with $\phi$ the normalized version of the “tent function”

$$\Phi_R(x) = A \cdot \begin{cases} 
1 - |x| & |x| \leq R \\
0 & |x| > R
\end{cases}.$$ 

(b) Noting the invariance of $H_\lambda$ under reflections ($x \rightarrow -x$), would you expect its ground state to be: odd, even, neither (under reflections).
(State the relevant general principle, in case one comes to mind.)

(c) Determine the operator’s ground state energy through the computation of the corresponding eigenfunction.

*Hint: the delta potential in the Schrödinger equation for the eigenfunction $\Psi(x)$ forces discontinuity in the derivative $\frac{\partial}{\partial x} \Psi(x)$ at $x = 0$.\*
Section B. Statistical Mechanics and Thermodynamics

1. Consider a 3D lattice occupying a volume $V$. On each of the $N$ sites, is placed a spin-$\frac{1}{2}$ with magnetic moment $2s_n\mu = \pm \mu \equiv \sigma \mu$, with $\sigma_n = 2s_n = \pm 1$.

In a magnetic field of strength $h$ the system’s Hamiltonian is taken to be

$$H_J = -\mu \sum_n \sigma_n h - J \sum_{\langle n, n' \rangle} \sigma_n \sigma_{n'} , \quad (1)$$

where $\sigma_n = \pm 1$ indexes the two spin states at site $n$, and the sum $\langle n, n' \rangle$ is over pairs of nearest neighbor spins, each pair counted once.

(a) First, for $J = 0$, evaluate the non-interacting system’s thermal expectation value of individual spins:

$$m_0(T, h) = \langle \sigma \rangle_{T, h} \quad \text{for } J = 0 , \quad (2)$$

*Hint: in the non-interacting case the thermal average $\langle \sigma \rangle_T$ can be equivalently evaluated by considering the ensemble of states of the entire system or of just the thermal average over two states of the single spin variable.*

(b) For the interacting system, with $J > 0$, relate the thermal average $m_J(T, h) = \langle \sigma \rangle_{T, h}$ to the magnetization per volume $M(T, h)$, and express it (either $M$ or $m$) in terms of the derivative in $h$ of the relevant thermodynamic function.

(c) Next consider the above ferromagnetic system ($J > 0$) in the mean-field approximation. Each spin interacts via $J$ with $z$ nearest neighbors. In this approximation the neighbors’ fluctuating values are replaced by their thermal average $m_J(T, h)$. Derive the corresponding self-consistency equation for $m_J(T, h)$.

*Hint: the relation can be presented in the form*

$$m = F(m, T, h) , \quad (3)$$

*where $m$ is an abbreviation for $m_J(T, h)$, and $F$ the function to be determined.*

(d) Within the above mean field approximation, explain the existence of a phase transition, at a critical temperature $T_c$ such that the residual magnetization at $h = 0$ satisfies

$$m_J(T, 0) = \lim_{h \to 0^+} m_J(T, h) \begin{cases} = 0 & T \leq T_c \\ > 0 & T > T_c \end{cases} \quad (4)$$

Determine the critical temperature $T_c$ (as function of $\mu, J, z$).
2. In the Joule-Thomson process interactions in a many-body system are exploited for refrigeration and liquefaction. The process is not effective when the working substance is an ideal gas.

\[
\begin{array}{cc}
V_1 & \text{Initial} \\
\hline
P_1 & \\
V_2 & \text{Final}
\end{array}
\]

A cylinder is divided by a fixed porous plug (shaded grey) into two chambers, labelled 1 and 2. Using a piston which is under constant pressure \( P_1 \) gas occupying an initial volume \( V_1 \) at temperature \( T_1 \) is forced through the pores of the plug into volume \( V_2 \) (right figure), which is maintained at a lower pressure \( P_2 \) (\(< \ P_1 \)). The process is adiabatic (no heat loss or gain through the walls).

The purpose of the following calculation is to determine the conditions under which the temperature at the second chamber \( T_2 \) will be lower than \( T_1 \).

(a) Write an expression for the work done on the gas as its state (parametrized by \((P, T)\)) changes from \((P_1, T_1)\) to \((P_2, T_2)\).

(b) Show that the enthalpy \( H = U + PV \) is conserved where \( U \) is the internal energy (recall the process is adiabatic).

In an ideal gas, \( T_2 = T_1 \) (see (e) below). In real gases, however, we have \( T_2 \neq T_1 \) because of interactions. The difference is expressed as the Joule-Thomson coefficient \( \mu \equiv (dT/dP)_H \) (subscript \( H \) means at constant enthalpy).

(c) Derive the general relation \( dH = TdS + VdP \).

(d) Derive an expression for \( \mu \) in terms of the easily measured quantities \( V, T, \partial V/\partial T \), and \( C_p \) (heat capacity at constant pressure). Start with part (c) and convert the differential \( dS \) to \( dT \) and \( dP \).

\((Hint: \text{ It may be helpful to recall the Maxwell relation } (\frac{\partial S}{\partial P})_T = - (\frac{\partial V}{\partial T})_P).\)

(e) Show that \( \mu = 0 \) for the ideal gas.
3. A spherical shell is floating in space filled with nitrogen gas ($N_2$). Gas is leaking from a punctured hole, which for simplicity we assume to be much smaller than the mean free path of the gas. Assume further that the escape rate is slow enough that the remaining gas remains in equilibrium by collisions among the gas molecules.

(a) What is the average energy of a gas molecule inside the cylinder when it is at temperature $T$? Treat the rotational modes of the diatomic gas classically, but neglect the vibrational mode which is not excited at room temperature.

(b) What is the average energy, including both kinetic and rotational, per molecule of gas that escapes when the gas inside the sphere is at temperature $T$?

(c) Suppose the sphere initially contains $N_0$ molecules at temperature $T_0$. Once there are only $N$ molecules remaining, what is the temperature $T$ of the remaining gas?

(d) Eventually all of the gas will escape. Assume that the sphere does not rotate, so that the leaking hole remains pointing in the same direction. What will the shell’s momentum be once all the gas is gone ($N = 0$)?

*Hint: In the last problem, in lieu of an exact calculation, credit will be given for a quicker order-of-magnitude estimate of the final momentum.*

You may find the following formula useful for your derivations:

$$
\int_0^\infty e^{-u} u^n du = \Gamma(n + 1) = \begin{cases} 
\frac{n!}{(2n+1)!} \sqrt{\frac{4\pi}{4n+1}} & \text{if } n \text{ is an integer} \\
\frac{(2n+1)!}{4n+1(n+1/2)!} & \text{if } n + \frac{1}{2} \text{ is an integer}
\end{cases}
$$