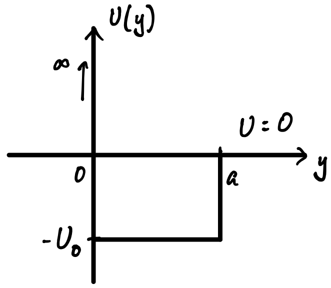
JI8T.3 (Particles on a Line)

(a)



ē Hamiltonian is $H = fx_N + \sum_{n=1}^{N}\left[\frac{p_n^2}{2m} + U(x_n - x_{n-1})\right]$

ē partit^{n} funct^{n} is then:

$$Q = \sum_{i_1} \cdots \sum_{i_N} \left(\prod_{i=1}^{N} e^{-\beta p_i^2/2m}\right)\left(\prod_{i=1}^{N} e^{-\beta U(x_i - x_{i-1})}\right)$$
$$\equiv \frac{1}{(2\pi\hbar)^N}\left[\int d^3p\; e^{-p_i^2/2m}\right]^N \underbrace{\left[\prod_{i=1}^{N} \int d^3x_i\; e^{-\beta U(x_i - x_{i-1})}\, e^{-\delta_{iN}\beta f x_N}\right]}_{A}$$
$$= \frac{1}{\lambda^N} A$$

Then, $\log Q = -N\log\lambda + \log A$

ē mean length is then: $\langle x_N\rangle = \frac{1}{Q}\sum_i x_{N_i} e^{-\beta e_i}$

$$= -\frac{1}{\beta}\frac{\partial \log Q}{\partial f} \quad , \because \text{only ē } x_N \text{ term has ē coefficient } f.$$
$$= -\frac{1}{\beta}\frac{\partial \log A}{\partial f} \quad , \because \lambda \text{ is independent of } f.$$

Now, we evaluate A explicitly:

A is ē product of N integrals, so just consider ē last one for now (ē integral over dx_N)

ē limits for x_N are x_{N-1} to infinity, so:

$$\int dx_N\; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$
$$\equiv \int_{x_{N-1}}^{\infty} dx_N\; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$
$$= \int_{x_{N-1}}^{x_{N-1}+a} dx_N\; e^{+\beta U_0 - \beta f x_N} + \int_{x_{N-1}+a}^{\infty} dx_N\; e^{-\beta f x_N} \quad (\because U(y) = 0 \text{ for } y > a)$$
$$= e^{+\beta U_0}\frac{1}{\beta f}\left(e^{-\beta f x_{N-1}} - e^{-\beta f x_{N-1} - \beta f a}\right) + \frac{1}{\beta f} e^{-\beta f x_{N-1} - \beta f a}$$
$$= \frac{1}{\beta f} e^{-\beta f x_{N-1}}\left[e^{-\beta f a} + e^{+\beta U_0}\left(1 - e^{-\beta f a}\right)\right]$$
$$\equiv \alpha e^{-\beta f x_{N-1}}, \quad w/\ \alpha = \frac{1}{\beta f}\left[e^{-\beta f a} + e^{+\beta U_0}\left(1 - e^{-\beta f a}\right)\right] \quad (\text{const.})$$

Then, ē next integral is (x_{N-1}):

$$\int_{x_{N-2}}^{\infty} dx_{N-1}\; \alpha e^{-\beta U(x_{N-1} - x_{N-2}) - \beta f x_{N-1}} \quad , \text{ which is identical to ē earlier integral!}$$
$$= \alpha^2 e^{-\beta f x_{N-2}}$$

Repeat^{g} this for all terms in A: $A = \alpha^N e^{-\beta f x_0} = \alpha^N$ $(\because x_0 := 0)$.

$$\Longrightarrow \langle x_N \rangle = -\frac{1}{\beta}\frac{\partial \log A}{\partial f}$$

$$= -\frac{N}{\beta}\frac{\partial \log \alpha}{\partial f}$$

$$\log \alpha = -\log(\beta f) + \log\left[e^{-\beta f a} + e^{+\beta U_0}\left(1 - e^{-\beta f a}\right)\right]$$

$$\Longrightarrow \frac{\partial \log \alpha}{\partial f} = -\frac{1}{f} + \left[e^{-\beta f a} + e^{+\beta U_0}\left(1 - e^{-\beta f a}\right)\right]^{-1}\left(-\beta a e^{-\beta f a} - \beta a e^{+\beta U_0 - \beta f a}\right)$$

$$= -\frac{1}{f} - \beta a e^{-\beta f a} \cdot \frac{1 + e^{+\beta U_0}}{e^{-\beta f a} + e^{-\beta U_0}\left(1 - e^{-\beta f a}\right)}$$

$$\therefore \langle x_N \rangle = \frac{N}{\beta f} + N a e^{-\beta f a} \cdot \frac{1 + e^{+\beta U_0}}{e^{-\beta f a} + e^{+\beta U_0}\left(1 - e^{-\beta f a}\right)}$$

(b)(i) For all $\bar{c}$ interparticle distances $(x_n - x_{n-1}) \gg a$, we revisit $\bar{c}$ integral:

$$\int dx_N \; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$

$$\equiv \int_{x_{N-1}}^{\infty} dx_N \; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$

$$\approx \int_{x_{N-1}+a}^{\infty} dx_N \; e^{-\beta f x_N}$$

$$= \frac{1}{\beta f} e^{-\beta f x_{N-1}} e^{-\beta f a} \equiv \gamma e^{-\beta f x_{N-1}}, \quad w/ \; \gamma = \frac{1}{\beta f} e^{-\beta f a}$$

$$\Longrightarrow A = \gamma^N$$

$$\text{So, } \langle x_N \rangle = -\frac{N}{\beta}\frac{\partial \log \gamma}{\partial f}$$

$$= -\frac{N}{\beta}\frac{\partial}{\partial f}\left[-\log(\beta f) - \beta f a\right]$$

$$= \frac{N}{\beta f} + Na \,/\!/.$$ (this is $\bar{c}$ high T or $\beta \to 0$ limit of part (a)).

(ii) For $(x_n - x_{n-1}) \ll a$, $\bar{c}$ integral is:

$$\int dx_N \; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$

$$\equiv \int_{x_{N-1}}^{\infty} dx_N \; e^{-\beta U(x_N - x_{N-1}) - \beta f x_N}$$

$$\approx \int_{x_{N-1}}^{x_{N-1}+a} dx_N \; e^{-\beta U_0 - \beta f x_N}$$

$$= e^{-\beta U_0}\frac{1}{\beta f}\left(e^{-\beta f x_{N-1}} - e^{-\beta f x_{N-1} - \beta f a}\right)$$

$$= \eta e^{-\beta f x_{N-1}}, \quad w/ \; \eta = \frac{1}{\beta f} e^{-\beta U_0}\left(1 - e^{-\beta f a}\right)$$

$$\Longrightarrow A = \eta^N$$

So, we get $\langle x_N \rangle = -\frac{N}{\beta}\frac{\partial \log \eta}{\partial f}$

$$= -\frac{N}{\beta}\frac{\partial}{\partial f}\left[-\log(\beta f) - \beta U_0 + \log\left(1 - e^{-\beta f a}\right)\right]$$

$$= \frac{N}{\beta f} + Na\frac{1}{e^{\beta f a} - 1}$$ (this also makes sense as it is $\bar{c}$ $\beta \to \infty$, high T limit of (a)).

(iii) Ngl, I don't know how to treat $\bar{c}$ integral for $\bar{c}$ case of a uniform distributⁿ over $(0, a)$.

However, this means each particle will be (on average) $\frac{a}{2}$ from $\bar{c}$ previous particle.

Thus, we can deduce ∄ $\langle x_N \rangle = \frac{Na}{2}$ //.