J18Q.3 (White Dwarf Star)

(a) We consider each e^- to be a fermion in a box of vol. $L^3 = V$.

Then, it can have wavevectors $\vec{k} = \frac{2\pi}{L}\langle n_x, n_y, n_z\rangle$, demand^{g} periodic boundary cond$^{it^n}$s.

$\bar{c}$ total no. of states w/ $|\vec{k}|$ up to a given k is: $N(k) = \frac{4/3\,\pi k^3}{(2\pi/L)^3} = \frac{Vk^3}{6\pi^2}$

$$\Longrightarrow N(\varepsilon) = \frac{V}{6\pi^2}\cdot\frac{2\sqrt{2}\,m^{3/2}}{\hbar^3}\,\varepsilon^{3/2}\cdot 2 \quad \text{(factor of 2 for spin degeneracy)}$$

$$= \frac{\sqrt{2}}{3}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\,\varepsilon^{3/2}\cdot 2$$

$$\Longrightarrow g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} = \sqrt{2}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\,\varepsilon^{1/2} \quad \text{(density of states)}$$

$\bar{c}$ Fermi energy ε_f is $\bar{c}$ energy of $\bar{c}$ highest occupied state at $T=0$.

Then, $\bar{c}$ total number of electrons is:

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon)\, n_{FD}(\varepsilon) \quad \text{, where } n_{FD} \text{ is } \bar{c} \text{ Fermi-Dirac distribut}^{n}$$

$$\equiv \int_0^{\varepsilon_f} d\varepsilon\cdot\sqrt{2}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\sqrt{\varepsilon} \quad \text{(where } n_{FD} \text{ becomes a step distribut}^{n} \text{ for } T=0\text{)}.$$

$$= \frac{2\sqrt{2}}{3}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\,\varepsilon_f^{3/2}$$

$$\Longrightarrow \varepsilon_f = \left(\frac{3\pi^2\hbar^3 N}{2\sqrt{2}\,Vm^{3/2}}\right)^{2/3}$$

Then, $\bar{c}$ ground state energy is:

$$U_0 = \int_0^{\varepsilon_f} d\varepsilon\cdot\varepsilon\, g(\varepsilon)$$

$$= \sqrt{2}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\int_0^{\varepsilon_f} d\varepsilon\,\varepsilon^{3/2}$$

$$= \frac{2\sqrt{2}}{5}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\left(\frac{3\pi^2\hbar^3 N}{2\sqrt{2}\,Vm^{3/2}}\right)^{5/3}$$

$$= \frac{2\sqrt{2}}{5}\,\frac{Vm^{3/2}}{\pi^2\hbar^3}\cdot\left(\frac{3\pi^2 N}{V}\right)^{5/3}\cdot\frac{\hbar^5}{4\sqrt{2}\,m^{5/2}}$$

$$= \frac{3^{5/3}}{10}\cdot V^{-2/3}m^{-1}\pi^{4/3}\hbar^2 N^{5/3} = \frac{\hbar^2}{10m}\left(\frac{3^5\pi^4 N^5}{V^2}\right)^{1/3}$$

Finally, invoke $V = \frac{4}{3}\pi R^3 \Longrightarrow \varepsilon_f = \left(\frac{9\pi\hbar^3 N}{8\sqrt{2}\,R^3 m^{3/2}}\right)^{2/3}$

$$\Longrightarrow U_0 = \frac{9\hbar^2}{10mR^2}\left(\frac{3\pi^2}{16}N^5\right)^{1/3} \quad \text{(correct dimens}^{n}\text{)}.$$

(b) $\bar{c}$ equilibrium radius is achieved when $\bar{c}$ two contribut^{n}s are comparable. (technically need to compute $\frac{\partial}{\partial r}(U_k + U_g)\big|_{r_0} = 0$). (virial thm).

$$\Longrightarrow \frac{3GM^2}{5R_0} = \frac{9\hbar^2}{10m_e R_0^2}\left(\frac{3\pi^2}{16}N^5\right)^{1/3}$$

$$R_0 = \frac{5}{3GM^2}\cdot\frac{9\hbar^2}{10m_e}\left(\frac{3\pi^2}{16}N^5\right)^{1/3} = \frac{3\hbar^2}{2GM^2 m_e}\left(\frac{3\pi^2}{16}\right)^{1/3}\cdot\frac{M^{5/3}}{m_p^{5/3}} \sim M^{-1/3}\text{ ✓}.$$

(c) For an ultrarelativistic Fermi gas, we use ē dispers^{n} relat^{n}: $E = |\vec{p}|c = \hbar ck$

$$\Longrightarrow N(\varepsilon) = \frac{V}{6\pi^2} \frac{\varepsilon^3}{\hbar^3 c^3}$$

$$\Longrightarrow g(\varepsilon) = \frac{V}{2\pi^2 \hbar^3 c^3} \varepsilon^2$$

Then, ē total number is: $N = \frac{V}{2\pi^2\hbar^3c^3} \int_0^{\varepsilon_f} d\varepsilon \, \varepsilon^2$

$$= \frac{V}{6\pi^2\hbar^3c^3} \varepsilon_f^3 \Longrightarrow \varepsilon_f = \left(\frac{6\pi^2\hbar^3c^3 N}{V}\right)^{1/3}$$

ē kinetic energy is: $U_0 = \frac{V}{2\pi^2\hbar^3c^3} \int_0^{\varepsilon_f} d\varepsilon \, \varepsilon^3$

$$= \frac{V}{8\pi^2\hbar^3c^3} \varepsilon_f^4$$

$$= \frac{V}{8\pi^2\hbar^3c^3} V^{-4/3} \cdot (6\pi^2 N)^{4/3} \hbar^4 c^4$$

$$= \frac{\hbar c}{8\pi^2 V^{1/3}} (6\pi^2 N)^{4/3}$$

$$= \frac{\hbar c}{8\pi^2 V^{1/3}} \left(\frac{6\pi^2 M}{m_p}\right)^{4/3}$$

$$= \frac{\hbar c}{8\pi^2 R} \left(\frac{6\pi^2 M}{m_p}\right)^{4/3} \left(\frac{3}{4\pi}\right)^{1/3}$$

(d) ē equilibrium radius here doesn't exist, so ē star must stay below ē mass limit to be stable.