10.2 J18M.3

a) After the collision, the center of mass is now $r_{\rm cm} = \alpha R/(1+\alpha)$ from the center of the planet toward the asteroid. Let $r_{\rm cm}$ be the new origin and define $\hat{\mathbf{x}}_3$ to point from the center of mass to the asteroid. Then, I_3 is unchanged and using the parallel axis theorem,

$$I_{(1,2)} = \frac{2}{5}MR^2 + Mr_{\rm cm}^2 + \alpha M(1 - r_{cm})^2$$

Therefore,

$$\boxed{I_{(1,2)} = \left(\frac{2}{5} + \frac{\alpha}{1+\alpha}\right)MR^2, \quad I_3 = \frac{2}{5}MR^2}$$

b) Let $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ be a set of unit vectors s.t. $\omega_0 = \omega_0\hat{\mathbf{k}}$. Since $\hat{\mathbf{k}} = \cos\theta\hat{\mathbf{x}}_3 - \sin\theta\hat{\mathbf{x}}_2$, the initial angular momentum is,

$$\mathbf{L}_0 = I_0\omega_0(\cos\theta\hat{\mathbf{x}}_3 - \sin\theta\hat{\mathbf{x}}_2)$$

where $I_0 = 2MR^2/5$. The asteroid collided with zero angular momentum with respect to the center of the planet, which implies that the motion was exactly radial with respect to the planet. Therefore, just after the collision, the conservation of angular momentum results in,

$$I_0\omega_0(\cos\theta\hat{\mathbf{x}}_3 - \sin\theta\hat{\mathbf{x}}_2) = I'\omega_1(0)\hat{\mathbf{x}}_1 + I'\omega_2(0)\hat{\mathbf{x}}_2 + I_0\omega_3(0)\hat{\mathbf{x}}_3$$

where $I' = [2/5 + \alpha/(1+\alpha)]MR^2$. Therefore the initial conditions are,

$$\omega_1(0) = 0, \ \omega_2(0) = -\frac{I_0\omega_0}{I'}\sin\theta, \ \omega_3(0) = \omega_0\cos\theta$$

which evolves following Euler's equations given by,

$$\begin{aligned} \dot{\omega}_1 - \gamma\omega_3\omega_2 &= 0 \\ \dot{\omega}_2 + \gamma\omega_3\omega_1 &= 0 \\ \dot{\omega}_3 &= 0 \end{aligned}$$

where $\gamma = 1 - I_0/I'$. Since ω_3 is constant, we can take the derivative of the first equation and substitute into the second to find,

$$\ddot{\omega}_1 = -\gamma^2\omega_3^2\omega_1$$

Therefore, the procession frequency is,

$$\dot{\phi} = \gamma\omega_3 = \gamma\omega_0\cos\theta$$

The procession period is then,

$$\boxed{T = \frac{2\pi}{\dot{\phi}} = \frac{T_0}{\gamma\cos\theta}}$$

where $T_0 = 2\pi/\omega_0$.