3. **Doomsday Scenario**

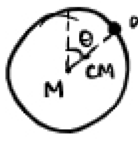
A unsuspecting solid spherical planet of mass $M$, radius $R$, and uniform mass density initially rotates with angular velocity $\omega_0$. Suddenly, an asteroid of mass $\alpha M$, incident with zero angular momentum about the center of the planet, smashes into and sticks to the planet at a location which is at polar angle $\theta$ relative to the initial rotational axis. Treat the asteroid as a point mass that sticks to the surface of the planet. The new mass distribution is no longer spherically symmetric, and as a result the rotational axis will in general process (the planet will also recoil, but this has no effect on the rotation questions we will pursue, so ignore the recoil). Recall Euler's equation for the time evolution of the angular momentum of a solid body in the absence of external torques (all quantities being expressed in a body-fixed coordinate system):

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = 0$$

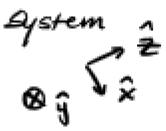
**(a)** What is the new moment of inertia tensor $I^{CM}_{\alpha\beta}$ about the new center of mass? Give it in terms of its principal axes in the body frame, and state clearly what these axes are. Recall $I = \frac{2}{5}MR^2$ for a uniform solid sphere.

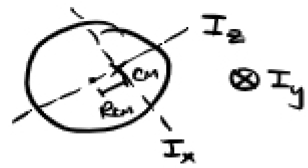
**(b)** What is the period of precession of the rotational axis in terms of the original rotation period $2\pi/\omega_0$?

(a) New center of mass moves slightly towards the asteroid




define coordinate system




where $(R_{CM})_z = \frac{M(0) + M\alpha(R)}{M(1+\alpha)} = \frac{\alpha}{1+\alpha}R$

For original CM we have $I_z = \frac{2}{5}MR^2$ and $I_{x,y} = \frac{2}{5}MR^2 + \alpha MR^2$

(for $I_z$, $\alpha M$ doesn't add an extra term because $\alpha M$ lies on the $z$ axis)

by parallel axis theorem, $I_z$ remains the same (the axis itself did not move)

but $I_x$ and $I_y$ both change because CM has moved, so at the sphere's center:

$$I_{\text{original center}} = I^{CM}_{x,y} + (M+\alpha M)\left(\frac{\alpha}{1+\alpha}R\right)^2$$

$$I_{x,y} = I^{CM}_{x,y} + \underbrace{(M+\alpha M)}_{\text{new total mass}}\underbrace{\left(\frac{\alpha}{1+\alpha}R\right)^2}_{\text{distance to new CM}}$$

$$\Rightarrow I^{CM}_{x,y} = I_{x,y} - (M+\alpha M)\left(\frac{\alpha}{1+\alpha}R\right)^2$$

$$= \frac{2}{5}MR^2 + \alpha MR^2 - M(1+\alpha)\left(\frac{\alpha^2}{(1+\alpha)^2}\right)R^2$$

$$= MR^2\left(\frac{2}{5} + \alpha - \frac{\alpha^2}{1+\alpha}\right) = MR^2\left(\frac{2}{5} + \frac{\alpha+\alpha^2-\alpha^2}{1+\alpha}\right)$$

$$= MR^2\left(\frac{2}{5} + \frac{\alpha}{1+\alpha}\right)$$

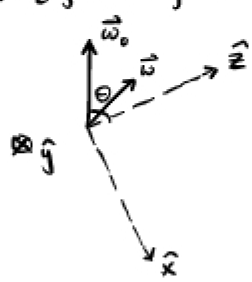
$$I^{CM}_{\alpha\beta} = \begin{pmatrix} \frac{2}{5}+\frac{\alpha}{1+\alpha} & 0 & 0 \\ 0 & \frac{2}{5}+\frac{\alpha}{1+\alpha} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} MR^2$$

(b) recall that $\frac{d\vec{L}}{dt} = \vec{\tau} = I^{cm}_{\alpha\beta}\,\dot{\vec{\omega}} = (I_x\dot\omega_x, I_y\dot\omega_y, I_z\dot\omega_z)$

$$\vec\omega \times \vec L = (\vec\omega \times I\vec\omega) = (\omega_x\hat x + \omega_y\hat y + \omega_z\hat z)\times(I_x\omega_x\hat x + I_y\omega_y\hat y + I_z\omega_z\hat z)$$
$$= \omega_x I_y\omega_y - \omega_x I_z\omega_z\hat y - \omega_y I_x\omega_x\hat z + \omega_y I_z\omega_z\hat x + \omega_z I_x\omega_x\hat y - \omega_z I_y\omega_y\hat x$$
$$= (I_z - I_y)\omega_y\omega_z\hat x + (I_x - I_z)\omega_x\omega_z\hat y + (I_y - I_x)\omega_x\omega_y\hat z$$

Gives us Euler's equations:

$$I_x\dot\omega_x = (I_z - I_y)\omega_y\omega_z$$
$$I_y\dot\omega_y = (I_x - I_z)\omega_x\omega_z$$
$$I_z\dot\omega_z = (I_y - I_x)\omega_x\omega_y$$




Find new $\vec\omega$ from these

First, notice $I_x = I_y$ so $I_z\dot\omega_z = 0 \Rightarrow \dot\omega_z = 0$ and $\omega_z(t)$ is a constant

$$\omega_z(t) = \omega_z(0) = \omega_0\cos\theta$$

↳ $\cos\theta$ because we expect $\omega_z = \omega_0$ at $\theta = 0$

Next, deal with other two equations

$$\dot\omega_x = \left(\frac{(I_z - I_y)\omega_z}{I_x}\right)\omega_y \qquad \dot\omega_y = \left(\frac{(I_x - I_z)\omega_z}{I_y}\right)\omega_x$$

since $I_x = I_y$ call both $I$ for now

$$\dot\omega_x = \underbrace{\left(\frac{-(I - I_z)}{I}\omega_z\right)}_{\text{Call this } -\Omega}\omega_y \qquad \dot\omega_y = \underbrace{\left(\frac{I - I_z}{I}\;\omega_z\right)}_{\text{Call this } \Omega}\omega_x$$

arbitrary choice of which is negative

$$\Rightarrow \dot\omega_x = -\Omega\omega_y \qquad \dot\omega_y = \Omega\omega_x$$
$$\Rightarrow \ddot\omega_x = -\Omega\dot\omega_y = -\Omega^2\omega_x$$

meaning the period of stable oscillations is $\Omega$

so new period is $T = \frac{2\pi}{\Omega} = \dfrac{2\pi}{\frac{I - I_z}{I}\omega_0\cos\theta}$

$$T = \frac{2\pi}{\omega_0}\left(\frac{I}{(I - I_z)\cos\theta}\right)$$

$$I = \left(\frac{2}{5} + \frac{\alpha}{1+\alpha}\right)MR^2 \qquad I_z = \frac{2}{5}MR^2$$

$$I - I_z = \frac{2}{5}MR^2 + \frac{\alpha}{1+\alpha}MR^2 - \frac{2}{5}MR^2 = \frac{\alpha}{1+\alpha}MR^2$$

$$\frac{I}{I - I_z} = \frac{\left(\frac{2}{5} + \frac{\alpha}{1+\alpha}\right)MR^2}{\frac{\alpha}{1+\alpha}MR^2} = \left(\frac{2}{5} + \frac{\alpha}{1+\alpha}\right)\left(\frac{1+\alpha}{\alpha}\right) = \frac{2(1+\alpha)}{5\alpha} + 1$$

$$= \frac{2 + 2\alpha + 5\alpha}{5\alpha} = \frac{2 + 7\alpha}{5\alpha}$$

$$T = \frac{2\pi}{\omega_0}\left(\frac{7\alpha + 2}{5\alpha\cos\theta}\right)$$