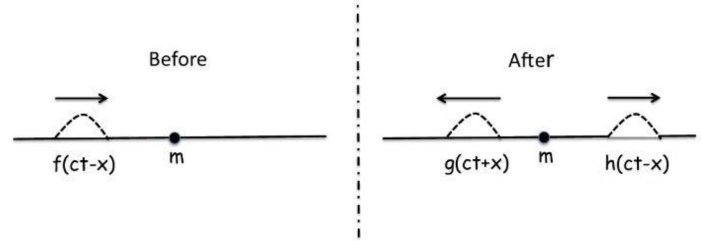
2. **Making Waves**



Small amplitude disturbances on an ideal string of linear mass density σ and tension τ are governed by a linear wave equation and move with sound speed c. A point particle of mass m is attached to the string at $x = 0$, as in the figure. Consider the scattering of a wave pulse from the mass: the string is initially at rest, and then a right-moving wave pulse $f(ct-x)$ is sent in from the left toward the mass. When this incident pulse hits the mass, a transmitted wave pulse $h(ct-x)$ (not identical in shape to the incident pulse) will be excited on the string to the right of the mass, and a reflected wave pulse $g(ct+x)$ will be excited to the left of the mass. In what follows, ignore gravity and stay non-relativistic.

(a) Write the wave equation for the string away from the point mass. What is the sound speed c in terms of the other parameters?

(b) Write the equation of motion for the mass on the string at $x = 0$. This equation serves as a (dynamical) boundary condition on the wave equation at this point.

(c) Solve for the reflected and transmitted wave functions g, h in terms of the incident wave function f. Hint: express the Fourier transforms of the outgoing wave pulses $\widetilde{g}(k)$ and $\widetilde{h}(k)$ in terms of the Fourier transform of the incident pulse $\widetilde{f}(k)$.

(d) Suppose a pulse containing only very long wavelength Fourier components is incident on the mass. In this long wavelength limit, what is the relation between the spatial profiles of the incident and reflected pulses (i.e. bring the solution back to x-space from Fourier space and "simplify" the expression in this limit)? What (roughly) do the wavelengths need to exceed for this result to be accurate?

a) $\sigma \frac{\partial^2 y}{\partial t^2} = F_T = \tau \frac{\partial^2 y}{\partial x^2}$

$\partial_t^2 y = c^2 \partial_x^2 y$

$c \overset{?}{=} \sqrt{\frac{\tau}{\sigma}}$ ← Tension

b) At the point w/ mass m,

The mass density has a spike so

$m \, \partial_t^2 y = \tau \left(\partial_x y|_{x=0^+} - \partial_x y|_{x=0^-} \right)$

at $x=0$ (Newton #2 Vertical)

Boundary condition is continuous in y, but not slope $\partial_x y$

$y(x=0_+) = y(x=0_-)$

c) $f = f(ct-x) = \int \tilde{f} \, e^{ik(ct-x)} \, dk$

$h = h(ct-x) = \int \hat{h} \, e^{ik(ct-x)} \, dk$

$g = g(ct+x) = \int \hat{g} \, e^{ik(ct+x)} \, dk$

$x<0, \quad y(x,t) = f(ct-x) + g(ct+x)$

$x>0, \quad y(x,t) = h(ct-x)$

$y'|_{0^-} = -f'(ct-x) + g'(ct+x)$

$y'|_{0^+} = -h'(ct-x)$

& $\quad f(ct) + g(ct) = h(ct)$

$\partial_t^2 y = \partial_t^2 (f+g)$ or $\partial_t h$ by continuity

$m\, \partial_t^2 y = \tau \left(\partial_x y|_{x=0^+} - \partial_x y|_{x=0^-} \right)$

$m\left(c^2 f'' + c^2 g''\right) = \tau\left(-h' + f' - g'\right)$

$ch' = cf' + cg'$ by continuity taking time derivatives

$m\left(c^2 f'' + c^2 g''\right) = \tau(-2g')$

$-f'' = g'' + \frac{2\tau}{mc^2} g'$

$k^2 \tilde{f} = \tilde{g}\left(-k^2 + ik\frac{2\tau}{mc^2}\right)$

$$\boxed{\begin{aligned} &\text{so } \frac{\tilde{g}}{\tilde{f}} = R(k) = \frac{-k^2}{k^2 - ik\, 2\tau/mc^2} \\ &\frac{h}{f} = 1 + \frac{g}{f} \\ &T = 1 + R = \frac{-ik\frac{2\tau}{mc^2}}{k^2 - ik\frac{2\tau}{mc^2}} \end{aligned}}$$

d) long k so $T \to 0, \quad R \to 1$

That means $f(ct) = g(ct)$

$R = -\frac{1}{1 - \underbrace{\frac{i\tau}{mc^2 k}}_{\text{small}}}$ so $1 \gg \frac{i\tau}{mc^2 k}$

$k \gg \frac{i\tau}{mc^2}$