Section A. Mechanics

1. **Stablized Instability**

In an ion trap, one uses electric fields to constrain the motion of an ion and, hopefully, confine it to some chosen region of empty space. Ideally, one would like to create an electrostatic potential $V(\vec{x})$ that has a minimum at $\vec{x} = 0$, but Poisson's equation $\nabla^2 V(\vec{x}) = 0$ means that this is not possible. The best one can do is to create a saddle: a potential that is stable about $\vec{x} = 0$ in some directions, and unstable in others, with the result that small perturbations will cause the ion to accelerate away in the unstable directions. However, if one creates a rotating potential so that in a non-inertial rotating frame the potential appears as time-independent, then motion about the origin can be made fully stable to small perturbations! Assume that the electronics of the trap have been arranged such that the ion (of mass m) sees the potential $V(x, y) = \frac{1}{2}\alpha(x^2 - y^2)$ in the non-inertial frame (x, y) that rotates about the z-axis with angular velocity Ω with respect to the inertial (lab) frame. In what follows, we study the motion of the ion in this rotating frame (ignoring motion in the direction orthogonal to the $x - y$ plane).

(a) In the rotating frame, write down the (Newtonian) equations of motion for the ion, including Coriolis and centrifugal terms. Note that these equations are linear in (x, y).

(b) The linear equations from part (a) have solutions that 'oscillate', i.e. have time dependence proportional to $e^{i\omega t}$. Derive the algebraic equation that determines the allowed (and in general complex) values of ω.

(c) Find the values of Ω for which all solutions for ω are real (so that the ion is stably trapped for all time within a region of space around the origin).

a) $\mathcal{L} = \frac{1}{2} m v^2$

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \Omega \times \vec{x}$$

$$\mathcal{L} = \frac{1}{2} m \left((v')^2 + 2 v' \cdot \Omega \times x' + (\Omega \times x')^2 \right)$$

A B C $x' \cdot (\Omega \times v') \times \Omega$

C A B

$x' \cdot \; v \times \Omega$

$$\frac{\partial \mathcal{L}}{\partial x'} = \frac{1}{2} m \left(2\, v \times \Omega + 2\, (\Omega \times x') \times \Omega \right)$$

$$\frac{\partial \mathcal{L}}{\partial v'} = \frac{1}{2} m \left(2 v' + 2 \Omega \times x' \right)$$

$$\frac{d}{dt} b = m \left(a' + \dot{\Omega} \times x' + \Omega \times v' \right)$$

$$m a' = - m \dot{\Omega} \times x' - m \Omega \times v' + m v' \times \Omega - m\, \Omega \times (\Omega \times x')$$

$$= - m \dot{\Omega} \times x' + 2 m v' \times \Omega - m\, \Omega \times (\Omega \times x')$$

$F_{accel} = 0$ F_{cor} F_{cent}

$$m \vec{a} = m \left(2 \vec{v} \times \Omega - \Omega \times (\Omega \times \vec{x}) \right)$$

b) $E = -\nabla V$ so $m \vec{a} = q \vec{E} = -q \vec{\nabla} V$ $\vec{\Omega} = w \hat{z}$

$$a = -\frac{q}{m} \left(\alpha x \hat{x} - \alpha y \hat{y} \right)$$

$$\vec{a} = 2 (v_x, v_y, 0) \times (0, 0, w) - (0, 0, w) \times \left[(0, 0, w) \times (x, y, 0) \right]$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & 0 \\ 0 & 0 & w \end{vmatrix} \qquad (-y w,\ x w,\ 0) \qquad \begin{vmatrix} x & y & z \\ 0 & 0 & w \\ x & y & 0 \end{vmatrix}$$

$$\vec{a} = 2 \left[v_y w \hat{x} - v_x w \hat{y} \right] - \left[-x w^2 \hat{x} - y w^2 \hat{y} \right] \qquad \begin{vmatrix} x & y & z \\ 0 & 0 & w \\ -yw & xw & 0 \end{vmatrix}$$

$$-\frac{q}{m} \alpha x = 2 \dot{y} w + x w^2 \qquad \dot{y} = \frac{-x}{2w} \left(w^2 + \frac{q}{m} \alpha \right)$$

$$-\frac{q}{m} \alpha y = -2 \dot{x} w + y w^2 \qquad \dot{x} = \frac{y}{2w} \left(w^2 + \frac{q}{m} \alpha \right)$$

$$\ddot{y} = -\frac{\dot{x}}{2w}\left(w^2 + \frac{g}{m}\alpha\right) = -\frac{y}{4w^2}\left(w^2 + \frac{g}{m}\alpha\right)^2$$

$$\ddot{y} = -\Omega^2 y$$

where $\Omega = \pm\sqrt{\frac{\left(w^2 + \frac{g}{m}\alpha\right)^2}{4w^2}}$

Same can be done for $\ddot{x}$

$$\ddot{x} = \frac{\dot{y}}{2w}\left(w^2 + \frac{g}{m}\alpha\right) = -\frac{x}{4w^2}\left(w^2 + \frac{g}{m}\alpha\right)^2 = -\Omega^2 x$$

c) Ω is always real. If $Im(\Omega) < 0$ stable

w can be complex... ? If $Im(\Omega) > 0$ unstable