10.3 J18E.3

a) Let $+\sigma$ plate be located on the $z = -d$ plane and the $-\sigma$ plate at $z = d$ plane. Using Gauss's law for a "Gaussian pillbox" volume saddling the positive plate, we find that $\mathbf{E} = \sigma/\epsilon_0\hat{\mathbf{z}}$ for $z \in (-d, d)$ and zero otherwise (which can be confirmed by again using a Gaussian pillbox volume containing both plates). From symmetry, we know that the force is in the z direction. Therefore, calculating T_{zz}, we find,

$$T_{zz} = \frac{1}{2}\epsilon_0 E^2 = \frac{\sigma^2}{2\epsilon_0}$$

To find the force on the positive plate, let us again use the Gaussian pillbox saddling the plate as our volume (call is Σ). Then,

$$\mathbf{F_A} = \oint_{\partial\Sigma} d\mathbf{a} \cdot \mathbf{T} = AT_{zz}\hat{\mathbf{z}} = \frac{A\sigma^2}{2\epsilon_0}$$

Therefore, the force per area on the plates are,

$$\boxed{\frac{\mathbf{F}_\pm}{A} = \pm\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}}}$$

which is towards each other as expected. This force is exactly half of $E\sigma = \sigma^2/\epsilon_0$ since the plate does not exert a force on itself (only the field from the other plate contributes).

b) Let the lines run along the z-axis with the $+\lambda$ line at $x = -d$ and the $-\lambda$ line at $x = d$. Denoting the electric field contribution of the $+\lambda$ line as E_+ and $-\lambda$ line as E_-, we find that,

$$\mathbf{E}_\pm = \pm\frac{\lambda}{2\pi\epsilon_0 r_\pm}\hat{\mathbf{r}}_\pm$$

where $r_\pm = |\mathbf{r} \mp d\hat{\mathbf{x}}|$ and $\hat{\mathbf{r}}_\pm$ is the radial vector with respect to the $\pm\lambda$ line. After some trigonometry, one can show that,

$$\hat{\mathbf{r}}_\pm = \frac{x \pm d}{r_\pm}\hat{\mathbf{x}} + \frac{y}{r_\pm}\hat{\mathbf{y}}$$

Therefore, the total electric field is,

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\lambda}{2\pi\epsilon_0}\left[\left(\frac{x+d}{r_+^2} - \frac{x-d}{r_-^2}\right)\hat{\mathbf{x}} + \left(\frac{y}{r_+^2} - \frac{y}{r_-^2}\right)\hat{\mathbf{y}}\right]$$

Since from symmetry we can see that the force must be along the x-axis, we want to evaluate T_{xx}, which results in,

$$T_{xx} = \frac{\lambda^2}{8\pi^2\epsilon_0}\left[\left(\frac{x+d}{r_+^2} - \frac{x-d}{r_-^2}\right)^2 - \left(\frac{y}{r_+^2} - \frac{y}{r_-^2}\right)^2\right]$$

To make the algebra easier, let us choose $x = 0$ as our surface — with the normal vector $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ (which gives us the force on the positively charged wire). Then, since $r(x=0)_\pm = \sqrt{d^2 + y^2}$, we find that,

$$T_{xx}(x=0) = \frac{\lambda^2}{2\pi^2\epsilon_0}\frac{d^2}{(d^2+y^2)^2}$$

Therefore, the force per unit length on the positive wire is,

$$\mathbf{f}_+ = \int_{-\infty}^{\infty} dy\ T_{xx}(x=0)\hat{\mathbf{x}} = \frac{\lambda^2}{2\pi^2\epsilon_0^2 d}\int_{-\infty}^{\infty} d\xi\ \frac{1}{(1+\xi^2)^2}\hat{\mathbf{x}}$$

The value of the integral is given in the question as $\pi/2$ (they actually provide the value of the integral over the half-line but the integrand is even so we can just double it). Thus,

$$\boxed{\mathbf{f}_\pm = \pm\frac{\lambda^2}{4\pi\epsilon_0^2 d}\hat{\mathbf{x}}}$$

where we used the fact that the force on the negative wire is equal and opposite. The force is attractive as expected.

c) If both lines are positively charged, the corresponding electric field becomes,

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\left[\left(\frac{x+d}{r_+^2} + \frac{x-d}{r_-^2}\right)\hat{\mathbf{x}} + \left(\frac{y}{r_+^2} + \frac{y}{r_-^2}\right)\hat{\mathbf{y}}\right]$$

then the stress tensor is,

$$T_{xx}(x=0) = -\frac{\lambda^2}{2\pi^2\epsilon_0}\frac{y^2}{(d^2+y^2)^2}$$

So the stress-energy tensor is now negative and vanishes at $y=0$ (directly between the two wires), unlike in the previous case. The force per unit length is then,

$$\mathbf{f}_\pm = \mp\frac{\lambda^2}{2\pi^2\epsilon_0 d}\int_{-\infty}^{\infty} d\xi\ \frac{\xi^2}{(1+\xi^2)^2}\hat{\mathbf{x}}$$

The integral can be evaluating by substituting $\xi = \tan\theta$,

$$\int_{-\infty}^{\infty} d\xi\ \frac{\xi^2}{(1+\xi^2)^2} = \int_{-\pi/2}^{\pi/2} d\theta \sin^2\theta = \frac{\pi}{2}$$

Thus, the force between the wires is given by,

$$\boxed{\mathbf{f}_\pm = \mp\frac{\lambda^2}{4\pi\epsilon_0^2 d}\hat{\mathbf{x}}}$$

which is repulsive as expected.