Section A. Mechanics

1. **Stablized Instability**

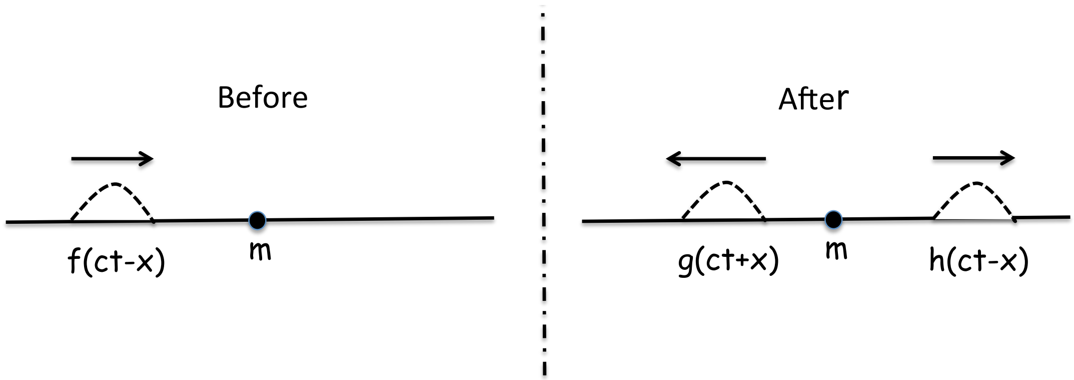
In an ion trap, one uses electric fields to constrain the motion of an ion and, hopefully, confine it to some chosen region of empty space. Ideally, one would like to create an electrostatic potential $V(\vec{x})$ that has a minimum at $\vec{x} = 0$, but Poisson's equation $\nabla^2 V(\vec{x}) = 0$ means that this is not possible. The best one can do is to create a saddle: a potential that is stable about $\vec{x} = 0$ in some directions, and unstable in others, with the result that small perturbations will cause the ion to accelerate away in the unstable directions. However, if one creates a rotating potential so that in a non-inertial rotating frame the potential appears as time-independent, then motion about the origin can be made fully stable to small perturbations! Assume that the electronics of the trap have been arranged such that the ion (of mass m) sees the potential $V(x, y) = \frac{1}{2}\alpha(x^2 - y^2)$ in the non-inertial frame (x, y) that rotates about the z-axis with angular velocity Ω with respect to the inertial (lab) frame. In what follows, we study the motion of the ion in this rotating frame (ignoring motion in the direction orthogonal to the $x - y$ plane).

(a) In the rotating frame, write down the (Newtonian) equations of motion for the ion, including Coriolis and centrifugal terms. Note that these equations are linear in (x, y).

(b) The linear equations from part (a) have solutions that 'oscillate', i.e. have time dependence proportional to $e^{i\omega t}$. Derive the algebraic equation that determines the allowed (and in general complex) values of ω.

(c) Find the values of Ω for which all solutions for ω are real (so that the ion is stably trapped for all time within a region of space around the origin).

2. **Making Waves**



Small amplitude disturbances on an ideal string of linear mass density σ and tension τ are governed by a linear wave equation and move with sound speed c. A point particle of mass m is attached to the string at $x = 0$, as in the figure. Consider the scattering of a wave pulse from the mass: the string is initially at rest, and then a right-moving wave pulse $f(ct - x)$ is sent in from the left toward the mass. When this incident pulse hits the mass, a transmitted wave pulse $h(ct - x)$ (not identical in shape to the incident pulse) will be excited on the string to the right of the mass, and a reflected wave pulse $g(ct + x)$ will be excited to the left of the mass. In what follows, ignore gravity and stay non-relativistic.

(a) Write the wave equation for the string away from the point mass. What is the sound speed c in terms of the other parameters?

(b) Write the equation of motion for the mass on the string at $x = 0$. This equation serves as a (dynamical) boundary condition on the wave equation at this point.

(c) Solve for the reflected and transmitted wave functions g, h in terms of the incident wave function f. Hint: express the Fourier transforms of the outgoing wave pulses $\widetilde{g}(k)$ and $\widetilde{h}(k)$ in terms of the Fourier transform of the incident pulse $\widetilde{f}(k)$.

(d) Suppose a pulse containing only very long wavelength Fourier components is incident on the mass. In this long wavelength limit, what is the relation between the spatial profiles of the incident and reflected pulses (i.e. bring the solution back to x-space from Fourier space and "simplify" the expression in this limit)? What (roughly) do the wavelengths need to exceed for this result to be accurate?

3. **Doomsday Scenario**

A unsuspecting solid spherical planet of mass M, radius R, and uniform mass density initially rotates with angular velocity ω_0. Suddenly, an asteroid of mass αM, incident with zero angular momentum about the center of the planet, smashes into and sticks to the planet at a location which is at polar angle θ relative to the initial rotational axis. Treat the asteroid as a point mass that sticks to the surface of the planet. The new mass distribution is no longer spherically symmetric, and as a result the rotational axis will in general precess (the planet will also recoil, but this has no effect on the rotation questions we will pursue, so ignore the recoil). Recall Euler's equation for the time evolution of the angular momentum of a solid body in the absence of external torques (all quantities being expressed in a body-fixed coordinate system):

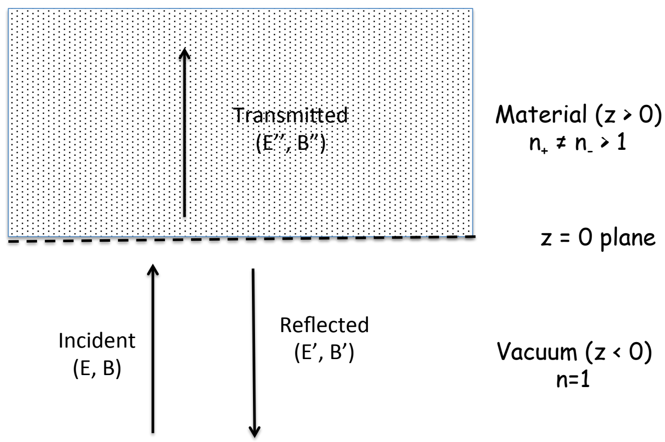
$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = 0$$

(a) What is the new moment of inertia tensor $I^{CM}_{\alpha\beta}$ about the new center of mass? Give it in terms of its principal axes in the body frame, and state clearly what these axes are. Recall $I = \frac{2}{5}MR^2$ for a uniform solid sphere.

(b) What is the period of precession of the rotational axis in terms of the original rotation period $2\pi/\omega_0$?

Section B. Electricity and Magnetism

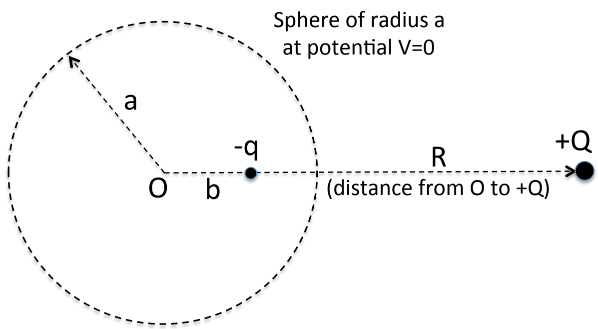
1. **Faraday Effect Polarizer**



In the figure, the region $z > 0$ is filled with a non-conducting ($\sigma = 0$) magnetically non-permeable ($\mu = 1$) material and the region $z < 0$ is vacuum. The material has different real indices of refraction $n_+(n_-)$ for right(left)-circularly polarized electromagnetic waves propagating in the z-direction. This 'Faraday effect', $n_+ \neq n_-$, can be arranged (in an appropriate material) by applying a magnetic field that points in the z-direction.

A linearly polarized plane wave is normally incident on the material from the vacuum. The resulting reflected and transmitted waves may be assumed to propagate, like the incident wave, in the z-direction. Calculate the ratio of the reflected to the incident intensity and quantitatively characterize the polarization of the reflected wave.

2. **Image Problems**



The 'method of images' allows us to solve many problems involving point charges and spherical conductors. In this question you are asked to obtain and then use this method in some representative applications:

(a) Consider a charge $+Q$ placed at a distance $R > a$ from the center of a sphere of radius a (for the moment, this is just a geometrical sphere, not a conductor or any other physical object). If a certain charge $-q$ of opposite sign is placed inside the sphere at the appropriate radius b (as in the figure) then the geometrical spherical surface of radius a is an equipotential surface of potential $V = 0$. What are q and b? You do not need to prove that this gives $V = 0$ over the full sphere, just get the correct image charge and position. This result can then be used in the following applications:

(b) A point charge Q is placed at a distance $R > a$ from the center of a conducting sphere of radius a. Find the force exerted on the sphere if the net total charge on the conducting sphere is Q.

(c) Consider the distribution of the total charge $Q \neq 0$ on the surface of the sphere under the conditions just described in (b). Find an equation for the largest distance R such that a zero surface charge density appears somewhere on the sphere.

(d) Two perfectly conducting spheres of radius a are placed far apart (their centers are separated by $R \gg 2a$) and kept at the same potential V_0 (this condition could be enforced by connecting the spheres with a fine wire). The full solution to this problem (which we are not asking you to do) involves an infinite sequence of image charges and image charges of image charges. For $R \gg 2a$, instead just include the leading image charge in each sphere and thus calculate the total charge on each sphere, image-corrected to leading order in the small parameter (a/R).

3. **Dealing with Stress**

The electromagnetic energy-momentum tensor $T_{\mu\nu}$ accounts for energy and momentum densities carried by the E and B fields as well as for pressure (and other stresses) associated with the presence of electric and magnetic fields. The spatial components of the tensor are given by

$$T_{ij} = \epsilon_0 \left(E_i E_j - \delta_{ij} \frac{E^2}{2} \right) + \frac{1}{\mu_0} \left(B_i B_j - \delta_{ij} \frac{B^2}{2} \right)$$

The net electromagnetic force on the charges and currents in some volume of space can be calculated as an integral of appropriate components of T_{ij} over the boundary of that volume. In answering the following questions, be sure to clearly state how you set up the surface integral of the stress tensor. Signs are important here!

(a) Consider the classic case of a parallel plate capacitor, with two infinite parallel plates separated by distance $2d$ and carrying a uniform surface charge density σ and $-\sigma$. Find the force per unit area exerted on the plates by the electric field using the energy-momentum tensor. Explain why the answer is not just equal to the field strength inside the capacitor times σ.

(b) Consider two infinite parallel lines of charge separated by distance $2d$ and carrying uniform linear charge densities λ and $-\lambda$. Find the force between the lines of charge using the energy-momentum tensor. (You might find the integral $\int_0^\infty dx/(1+x^2)^2 = \pi/4$ useful).

(c) Now let the two line charges carry the same charge density λ. In what way does the energy-momentum tensor calculation of the force change, and what is the final result?

Section A. Quantum Mechanics

1. **Two Hydrogen Atoms**

Consider two hydrogen atoms with their nuclei held at fixed positions such that the displacement $\vec{R}$ between them is large compared to the ground-state size of the atoms. Treat the Coulomb interaction as instantaneous (no retardation), neglect nuclear spins and nuclear motion, and neglect any contribution of the electron spins to the energy (no spin-orbit interactions). The appropriate basis set for discussing the energetics of this system is spanned by states of the type $\psi^{(1)}_{n\ell m}(\vec{r}_1)\psi^{(2)}_{n'\ell'm'}(\vec{r}_2)$ where each electron is in a single-atom eigenstate about its nucleus. Take the z axis to lie along the displacement $\vec{R}$ between the two nuclei. Under these conditions, the dominant interaction between the two atoms comes from the electric dipole-dipole interaction:

$$U_{dipole}(\vec{R}) = \frac{1}{4\pi\epsilon_0 R^3}\left[\vec{d}^{(1)} \cdot \vec{d}^{(2)} - 3d_z^{(1)}d_z^{(2)}\right]$$

where $\vec{d}^{(n)}$ is the dipole operator for the electron bound to nucleus n. The ground state energy of this pair of atoms depends on R at large R as

$$E_0(r) = E_0(\infty) + A_0 R^{-\delta_0} + \ldots .$$

The two electrons can of course be in either a state of total spin $S = 0$ or $S = 1$; in what follows, let the two electrons always have total spin $S = 0$.

(a) Write down the leading approximation to the ground state wave function in the limit of large R (i.e. explain how you would calculate $E_0(\infty)$).

(b) Explain what physics determines the exponent δ_0 governing the R-dependence of the correction to the ground state energy. Give the value of δ_0.

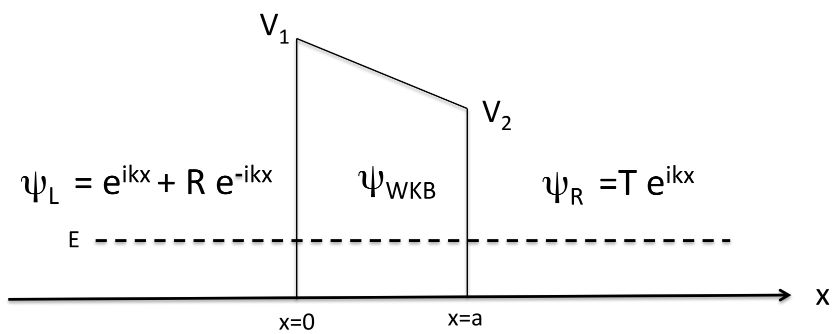
(c) Give an order of magnitude estimate for the constant A_0 and give a general argument for its sign.

(d) The energy of the lowest energy electronic *excited* eigenstate of this pair of atoms depends on R at large R as

$$E_1(R) = E_1(\infty) + A_1 R^{-\delta_1} + \ldots .$$

What is the exponent δ_1? Give an order of magnitude estimate of A_1. What is the sign of A_1? What is the full wavefunction of this excited eigenstate at leading order in large R?

2. **Tunneling Under A Sloping Barrier**



Consider an electron of energy E moving in one dimension and tunneling from left to right under the potential barrier shown in the figure. The potential has a constant slope $dV/dx = -(V_1 - V_2)/a$ in the barrier region (as might be the case in an electric dipole layer). The electron wave function for $x < 0$ is $\psi_L = e^{ikx} + Re^{-ikx}$ and the wave function for $x > a$ is $\psi_R = Te^{ikx}$. You are to calculate T, but to do that you need to know $\psi(x)$ in the classically forbidden region $0 < x < a$. For the purposes of this problem, let us assume that the WKB approximation to $\psi(x)$ is valid everywhere under the barrier. (NB: This means that the WKB wave vector $\kappa(x) = \sqrt{2m(V(x) - E)/\hbar^2}$ satisfies $\frac{1}{\kappa}\frac{d\kappa}{dx} \ll \kappa$ everywhere under the barrier.)

(a) Write down the WKB wave function in the barrier region. Remember that there are two WKB solutions, and they will both be present with unknown amplitudes. At this stage leave any WKB integrals as implicit functions of x.

(b) To verify that you have the correct form of the WKB approximation, calculate the probability flux in the barrier region for the general case where the two amplitudes are complex. Use any simplifications that follow from the WKB validity condition $\frac{d}{dx} log\kappa(x) \ll \kappa(x)$. Is the flux independent of x? If not, revisit part (a).

(c) The WKB wave function has to join correctly at $x = 0$ and at $x = a$ to the exterior wave functions ψ_L, ψ_R. Write down the two sets of matching conditions. Once again, use any simplifications that follow from from the WKB validity condition $\frac{d}{dx} log\kappa(x) \ll \kappa(x)$.

(d) Now also assume that the barrier is wide enough so that the WKB integral $\int_0^a dx\kappa(x) \gg 1$. Use this as needed to simplify the algebra and thus solve for the leading behavior of the transmission coefficient T in this limit.

3. **White Dwarf Star**

Model a white dwarf star as a degenerate non-interacting Fermi gas of electrons, supported against gravitational collapse by the electron degeneracy pressure. For simplicity, assume that the star is a sphere of radius R and *uniform* mass density, containing N electrons of mass m_e and N protons of mass $m_p \gg m_e$ for an approximate total mass of $M = Nm_p$.

(a) First, assume that the electrons are non-relativistic. Find their Fermi energy and thus calculate their contribution to the star's total ground-state kinetic energy.

(b) The gravitational potential energy of a uniform density sphere is

$$U_{grav} = -\frac{3GM^2}{5R} \ .$$

Find the equilibrium radius R for the ground state of this white dwarf, neglecting the protons' kinetic energy. How does this radius depend on the mass M? At roughly what mass M do the electrons in this ground state become relativistic?

(c) If instead the electrons are highly relativistic, so that their mass can be neglected, what is their Fermi energy and their total ground-state kinetic energy?

(d) Under what conditions is this highly-relativistic degenerate electron star unstable to gravitational collapse?

[This (d) is called the Chandrasehkar limit. A star that violates the limit will collapse into a neutron star or a black hole, depending on whether or not the neutron degeneracy pressure can support the star.]

Section B. Statistical Mechanics and Thermodynamics

1. **Particles in a Box**

First consider a single free non-relativistic particle of mass m confined to a three-dimensional volume V. The particle has no internal degrees of freedom. Let $Z_1(mT)$ denote the quantum partition function for this system at temperature T (where the partition sum is taken over the discrete energy levels of this system).

(a) Working in the classical (high temperature) regime, show that $Z_1(mT) \approx V/\lambda^3$ with a suitably defined de Broglie wavelength $\lambda(mT)$. Use this result to obtain the classical energy and heat capacity at fixed volume of this single-particle system.

(b) What (roughly) is the temperature at which this classical approximation breaks down?

(c) Now consider a system consisting of two identical, non-interacting such quantum particles in the same box. The particles either have no internal degrees of freedom in the case of bosons, or these degrees of freedom are held fixed in the case of fermions (e.g. both spins "up"). The two-particle partition function $Z_2(mT)$ contains the effects of identical-particle statistics. Show that the exact free boson and free fermion two-particle quantum partition functions $Z_2(mT)$ can in fact be expressed in a simple way (at all T) in terms of the exact one-particle quantum partition functions $Z_1(mT)$ and $Z_1(mT/2)$.

(d) Using the classical approximation $Z_1 = V/\lambda^3$ derived in the first part of this problem, calculate the leading high-temperature contribution to the energy E and the heat capacity C_V due to Bose or Fermi statistics in this two-particle system.

2. **Hawking radiation**

Hawking and Bekenstein famously proposed that a black hole is in fact a thermodynamic system with a thermodynamic energy equal to its mass energy (Mc^2) and also endowed with a temperature $T(M)$ and an entropy $S(M)$. The mass can be equivalently expressed in terms of the Schwarzschild radius $R = 2GM/c^2$ (G is Newton's constant) of the black hole's spherical event horizon. Hawking further established that the entropy equals exactly one-quarter the area of the horizon (measured in Planck units), a relation that can be written as

$$S/k_B = \frac{1}{4}\frac{4\pi R^2}{\hbar G/c} = \frac{4\pi G M^2}{\hbar c} \tag{1}$$

In what follows, we will work out some consequences of the thermodynamic nature of the black hole.

a) In thermodynamics, the temperature of a system is related to the derivative of the entropy with respect to the energy. Use this basic relation to calculate the temperature of the black hole event horizon for a black hole of mass M.

b) Assuming that the black hole only emits photons with a black-body radiation spectrum from the surface of the horizon which is at this temperature, find the rate of energy loss. Express your answer in terms of fundamental constants. You might find useful the integral $\int_0^\infty x^3 dx/(e^x - 1) = \pi^4/15$.

c) Find the time it takes a black hole of mass M to evaporate due to this energy loss (thus mass loss).

3. **Particles on a Line**

Consider a system of N classical particles on a line restricted to positions

$$0 < x_1 < x_2 < ... < x_{N-1} < x_N \ .$$

The Hamiltonian is

$$H = fx_N + \sum_{n=1}^{N}(\frac{p_n^2}{2m} + U(x_n - x_{n-1})) \ ,$$

so there is a compressive force $f > 0$. We define $x_0 = 0$. The potential between neighboring particles vanishes for $x_n \geq (x_{n-1} + a)$, so $U(y) = 0$ for $y \geq a > 0$, where $y = x_n - x_{n-1}$. At shorter distances the potential is attractive and constant, so $U(y) = -U < 0$ for $0 < y < a$. Note that the compressive force may equivalently be viewed as an additional linear term $+fy$ in the potential.

The two parts of this problem may be done independently, although of course their correct answers are consistent with each other:

(a) Compute the mean length, $\langle x_N \rangle$, of this system at thermal equilibrium, as a function of the temperature T and the given parameters.

(b) There are three limiting regimes where this result simplifies:

(i) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) \gg a$,

(ii) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) \ll a$,

(iii) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) < a$ but they are fairly uniformly distributed in $0 < (x_n - x_{n-1}) < a$.

State what the values of the parameters must be to be in each of these three simplifying limits, and what is $\langle x_N \rangle$ in each of these regimes.