J17Q.2 (s-wave Scatter^g Resonances)

$\bar{c}$ potential is given as: $V(r) = -V_0\,\theta(R-r)$

We know $\#$ at large distances from $\bar{c}$ potential ($r \gg R$), we must have:

$\psi(r) \cong Ae^{ikz} + B\frac{e^{ikr}}{r}$

Just outside $\bar{c}$ potential ($r \gtrsim R$), we have $\psi_{out} \approx A + \frac{B}{r}$

Inside $\bar{c}$ well, we want to solve $\bar{c}$ Schrödinger equat^n: $-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \left(\frac{\hbar^2 l(l+1)}{2mr^2} - V_0\right)\psi = E\psi$

W/ $\bar{c}$ ansatz $\psi(r,\theta) = R(r)f(\theta,\phi)$, & $u(r) = rR(r)$, this becomes:

$$-\frac{\hbar^2}{2m}u'' + \left(\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - V_0\right)u = Eu$$

For low energy (s-wave) scatter^g, we only need $\bar{c}$ $l=0$ component, so defin^g $k = \frac{\sqrt{2mV_0}}{\hbar}$, we have: $(E \approx 0)$.

$u'' = -k^2 u \implies u = \tilde{A}\sin(kr) + \tilde{B}\cos(kr) \implies R(r) = \tilde{A}\frac{\sin(kr)}{r} + \cancel{\tilde{B}\frac{\cos(kr)}{r}}$ (for normalisability)

Match^g $\bar{c}$ boundary condit^{n}s: ① Continuity of ψ: $\tilde{A}\frac{\sin(ka)}{R} = A + \frac{B}{R}$

② Continuity of ψ': $\tilde{A}\left(\frac{k\cos(ka)}{R} - \frac{\sin(ka)}{R^2}\right) = -\frac{B}{R^2}$

From ②: $-B = \tilde{A}\left[kR\cos(kR) - \sin(kR)\right]$

Into ①: $-B\frac{\sin(kR)}{R} = \left(A + \frac{B}{R}\right)\left(kR\cos(kR) - \sin(kR)\right)$

$$-B\frac{\sin(kR)}{R} = A\left(kR\cos(kR) - \sin(kR)\right) + B\left(k\cos(kR) - \frac{\sin(kR)}{R}\right)$$

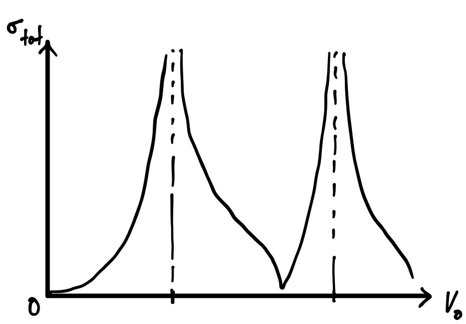
$$\implies \sin(kR) - kR\cos(kR) = \frac{B}{A}k\cos(kR)$$

$$\implies \tan(kR) - kR = \frac{B}{A}kR \implies \frac{B}{A} = \frac{1}{k}\tan(kR) - R$$

Then, we can obtain $\bar{c}$ differential cross sect^n as: $\frac{d\sigma}{d\Omega} = \left|\frac{B}{A}\right|^2 = \left|\frac{1}{k}\tan(kR) - R\right|^2$

$$= R^2\left(\frac{\tan(kR)}{kR} - 1\right)^2$$

This gives $\bar{c}$ total cross sect^n as: $\sigma_{tot} = \int d\Omega\,\frac{d\sigma}{d\Omega} = 4\pi R^2\left(\frac{\tan(kR)}{kR} - 1\right)^2 = 4\pi R^2\left(\frac{\tan(R\sqrt{2mV_0}/\hbar)}{R\sqrt{2mV_0}/\hbar} - 1\right)^2$



$\bar{c}$ cross sect^n diverges for all V_0 where $\frac{R\sqrt{2mV_0}}{\hbar} = \frac{2n+1}{2}\pi$, $n \in \mathbb{Z}$.

These are $\bar{c}$ points at which new bound states are admitted.