2 Jan 2022

Section A. Quantum Mechanics

1. **Interacting spins**

Four distinguishable spin-1/2 objects interact. The Hamiltonian is

$$H = A\vec{\sigma}_1 \cdot \vec{\sigma}_2 + B\vec{\sigma}_3 \cdot \vec{\sigma}_4 + C(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{\sigma}_3 + \vec{\sigma}_4) \ ,$$

where $\vec{\sigma}_n$ are the Pauli spin operators for spin n.

(a) List a complete set of good quantum numbers for the eigenstates of this Hamiltonian.

(b) List the eigenenergies and the degeneracy of each energy level.

(c) Using the basis of the eigenstates of the z-components of each spin, show a **ground state** for the case $A = B < 0 < C$. For this state give its complete set of good quantum numbers.

J17Q.1

a) $H = A(\sigma_1\cdot\sigma_2) + B(\sigma_3\cdot\sigma_4) + C(\sigma_1+\sigma_2)\cdot(\sigma_3+\sigma_4)$

Usual quantum numbers: $s_1, s_2, s_3, s_4, m_1, m_2, m_3, m_4$

m quantum numbers are now bad since multiple spins are coupled.

Move to basis with coupled spins $\rightarrow$

$\sigma_i + \sigma_j = \sigma_{ij}$, $\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 + 2\sigma_i\cdot\sigma_j \rightarrow \sigma_i\cdot\sigma_j = \frac{1}{2}(\sigma_{ij}^2 - \sigma_i^2 - \sigma_j^2)$; $\sigma_i^2|\psi\rangle = 4s_i(s_i+1)|\psi\rangle$

$$H = \frac{A}{2}(\sigma_{12}^2 - \sigma_1^2 - \sigma_2^2) + \frac{B}{2}(\sigma_{34}^2 - \sigma_3^2 - \sigma_4^2) + \frac{C}{2}(\sigma_{1234}^2 - \sigma_{12}^2 - \sigma_{34}^2)$$

$\boxed{\text{New good quantum numbers: } s_1, s_2, s_3, s_4, s_{12}, s_{34}, s_{1234}, m_{1234}}$

b) $s_1 = s_2 = s_3 = s_4 = \frac{1}{2}$, $4s_i(s_i+1) = 3$

$$H|\psi\rangle = \frac{A}{2}(4s_{12}(s_{12}+1) - 6) + \frac{B}{2}(4s_{34}(s_{34}+1) - 6)$$
$$+ \frac{C}{2}(4s_{1234}(s_{1234}+1) - 4s_{12}(s_{12}+1) - 4s_{34}(s_{34}+1))$$ $\leftarrow$ no m_{1234} dependence

s_{12}	s_{34}	s_{1234}	$H\|\psi\rangle$	$d = 2s_{1234}+1$
0	0	0	$-3A-3B$	1
0	1	1	$-3A+B$	3
1	0	1	$A-3B$	3
1	1	0	$A+B-8C$	1
1	1	1	$A+B-4C$	3
1	1	2	$A+B+4C$	5

c) $E_{gs} = A+B-8C$, $\boxed{S_{12}=1,\ S_{34}=1,\ S_{1234}=0,\ M_{1234}=0,\ S_1=S_2=S_3=S_4=\frac{1}{2}}$

Clebsch-Gordan for 1×1 (can be derived with lowering operator)

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|+1,-1\rangle - |0,0\rangle + |-1,+1\rangle\right)$$

$$|+1,-1\rangle = |\uparrow\uparrow\downarrow\downarrow\rangle;\quad |-1,+1\rangle = |\downarrow\downarrow\uparrow\uparrow\rangle$$

$$|0,0\rangle = \frac{1}{2}\left(|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle\right)$$

$$= \boxed{\frac{1}{\sqrt{3}}\left[|\uparrow\uparrow\downarrow\downarrow\rangle - \frac{1}{2}\left(|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle\right) + |\downarrow\downarrow\uparrow\uparrow\rangle\right]}$$