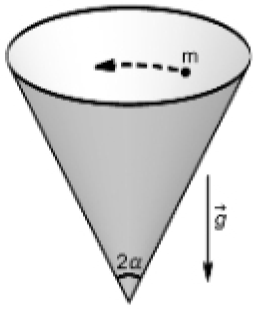
3. **Orbits in a cone**



A cone sits apex down with its axis vertical. The inside opening angle of the cone is 2α, with $\pi > 2\alpha > 0$, so the inside surface of the cone in cylindrical coordinates is $r = z\tan\alpha > 0$. A point mass slides without friction and nonrelativistically on the inside surface of the cone.

(a) Solve for the circular orbit with radius r_0. What is the period $T_0(r_0)$ of this orbit?

(b) If this circular orbit is slightly perturbed away from circular, what is the period $T_1(r_0)$ of the resulting oscillations in the radial position r? Show your work.

(a) For a fixed z, reduces to normal orbit w/

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2)$$

but additional gravitational potential energy from height z

$$\Rightarrow \mathcal{L} = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2) - mgz$$

but we know $r = z\tan\alpha \Rightarrow z = r/\tan\alpha$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mg\left(\frac{r}{\tan\alpha}\right)$$

E-L equations

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{mg}{\tan\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\Rightarrow m\ddot{r} = mr\dot{\phi}^2 - \frac{mg}{\tan\alpha}$$

for circular orbit $\ddot{r} = 0$

$$\Rightarrow mr\dot{\phi}^2 - \frac{mg}{\tan\alpha} = 0 \Rightarrow r\dot{\phi}^2 = \frac{g}{\tan\alpha}$$

$\dot\phi = \sqrt{\dfrac{g}{r\tan\alpha}}$ Using $T = \dfrac{2\pi}{\dot\phi}$ we get

$$T_0(r_0) = 2\pi\sqrt{\frac{r_0\tan\alpha}{g}}$$

(b) slight perturbation from circular $\Rightarrow$ $r = r_0 + \delta$ w/ $\delta \ll r_0$

plugging this into E-L eqs from (a) (w/ $\ddot r = \ddot\delta$)
Also need to use angular momentum l (a conserved quantity) rather than $\dot\phi$

$$l = mr^2\dot\phi$$
$$\Rightarrow \frac{l}{mr^2} = \dot\phi$$

then $m\ddot r = mr\dfrac{l^2}{m^2r^4} - \dfrac{mg}{\tan\alpha}$

$$\Rightarrow m\ddot r = \frac{l^2}{mr^3} - \frac{mg}{\tan\alpha}$$

$$m\ddot\delta = \frac{l^2}{m(r_0+\delta)^3} - \frac{mg}{\tan\alpha}$$

$$m\ddot\delta = \frac{l^2}{mr_0^3(1+\delta/r_0)^3} - \frac{mg}{\tan\alpha}$$

$\dfrac{\delta}{r_0} \ll 1 \Rightarrow$ Taylor expand $\dfrac{1}{(1+\delta/r_0)^3} \approx 1 - 3\left(\dfrac{\delta}{r_0}\right)$

$$m\ddot\delta = \frac{l^2}{mr_0^3}\left(1 - 3\frac{\delta}{r_0}\right) - \frac{mg}{\tan\alpha}$$

$$m\ddot\delta = \frac{l^2}{mr_0^3} - 3\left(\frac{l^2}{mr_0^4}\right)\delta - \frac{mg}{\tan\alpha}$$

these terms together $= 0$ from (a)

$$\Rightarrow \quad \ddot{\delta} = -3 \frac{l^2}{m^2 r_0^4} \delta$$

stable oscillations b/c coefficient < 0

$$\Rightarrow \quad \omega_{osc} = \sqrt{\frac{3l^2}{m^2 r_0^4}}$$

then plugging in value of l for circular orbit
→ w/ the understanding that it is a constant value even outside circular orbit case

$$l = m r_0^2 \dot{\phi} = m r_0^2 \sqrt{\frac{g}{r_0 \tan\alpha}}$$

$$\Rightarrow l^2 = m^2 r_0^4 \left(\frac{g}{r_0 \tan\alpha}\right) = m^2 r_0^3 g \left(\frac{1}{\tan\alpha}\right)$$

$$\frac{3l^2}{m^2 r_0^4} = \frac{3 m^2 r_0^3 g}{m^2 r_0^4 \tan\alpha} = \frac{3g}{r_0 \tan\alpha} = \omega_{osc}^2$$

then period of oscillations in r given by

$$T_1(r_0) = \frac{2\pi}{\omega_{osc}} = 2\pi \sqrt{\frac{r_0 \tan\alpha}{3g}}$$